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A Contribution to the Theory of Ferromagnetic Crystals

G. S. Mahajani

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II. *A Contribution to the Theory of Ferromagnetic Crystals.*

By G. S. MAHAJANI (*Fergusson College, Poona, and St. John's College, Cambridge*).

(*Communicated by* R. H. FOWLER, *F.R.S.*)

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Introduction.

THIS paper aims, in general, at giving an atomistic theory for magnetic phenomena in ferromagnetic crystals. In particular, it explains theoretically WEBSTER'S experimental results with regard to iron crystals.

As has been pointed out in the abstract,*, the fundamental fact of interest in experiments on magnetic substances is that the direction ψ of magnetization I does not in general coincide with the direction ϕ of the external field H. Experiments on iron crystals have now revealed that this deviation-effect $(\phi - \psi)$ exists even in crystals of cubic structure and a symmetrical boundary. In other words, cubical symmetry does not make a body magnetically isotropic, as was erroneously assumed by FARADAY, TYNDALL and PLÜCKER. To account for this deviation, and in general to relate the two directions ϕ, ψ, with each other must certainly be the object of any theory bearing on this subject. Accordingly, WEISS developed a macroscopic theory by introducing the conception of the molecular field. This idea served many useful purposes, especially in connection with pyrrhotite. But as is shown in the present paper, it is not adequate to explain the phenomena in the case of crystals with a cubic structure.

With respect to the question of the molecular field in iron, the position as summarised by KUNZ in 1922, was that its existence had not been definitely established. WEBSTER'S experiments mark an advance in the sense that not only do they establish the existence of the molecular field in iron, but they also give the order of its magnitude. He has had to give up, however, WEISS'S law of simple proportionality, and replace it by what may be called a fourth-power law,—which, as will be shown, is not acceptable on physical grounds.

It seemed better to give up any *a priori* ideas of the molecular field, and start right from the beginning with atomic actions. This alternative was conceived though not followed up by both WEBER and MAXWELL. " The molecules do not turn their axes parallel to x, and this is either because each molecule is acted on by a force tending to

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* " A contribution to the Theory of Ferromagnetism," ' Proc. Camb. Phil. Soc.,' vol. 23, p. 136 (1926).

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preserve it in its original position, or because an equivalent effect is produced by the mutual actions of the entire system of molecules. WEBER adopts the former of these suppositions as the simplest . . .''*

Now it may be that apart from the external field H, the internal field which governs the magnetization I of a body arises partly from the mutual action of the molecular magnets and partly from some other force of an obscure origin. The rational way of proceeding, however, is to try and see if we cannot do with the mutual actions only without assuming the existence of an extra force. This is the view-point adopted in this paper. The subject-matter, the theory, and the principal achievements thereof may be now briefly indicated.

Two types of ferromagnetic crystals are considered: 1, Iron, which has a cubic structure; and 2, Pyrrhotite which has a non-cubic structure. The ultimate particles that form the lattice-points of the structure are supposed to derive their magnetic properties from the presence of permanent electronic orbits. A parameter a is introduced to denote the mean radius of an orbit. The energy due to the mutual actions of the particles is calculated as a series of terms involving third, fifth, seventh powers of i/s, where s is the lattice-constant. The parameter a is also naturally involved. The *total* energy of the system is then determined per unit volume, in an external field H, and this energy density is made stationary to search for the steady states of magnetization. Temperature effects are left out of account, so that strictly the theory is a theory of the magnetic properties of these crystals near the absolute zero.

The first part deals with cubic structures in saturation states; and is devoted mainly to the consideration of WEBSTER'S results. It is shown that the deviation-effect can be accounted for both quantitatively and qualitatively by retaining the fifth-order terms in the energy function.† DE WAARD'S suggestion to identify each particle at a lattice-point with a simple magnetic-doublet is shown to be inadequate, as it leads to an energy function involving terms up to the third order only.

The second chapter discusses the position of WEISS'S molecular field in the present theory. His hypothesis of simple proportionality between the molecular field H_m and the magnetization I is generalized to cover the case of cubic structures. WEBSTER'S empirical suggestion to adopt a fourth-power relation is rejected on theoretical grounds; and it is shown that the *effective* molecular field obeys the third-power law.

The third chapter is devoted to numerical tests. Consideration of WEBSTER'S curves in the light of the present theory gives a to be of the order 10^{-9}. The effective molecular field which replaces the atomic actions is also shown to be of the same order of magnitude as found by WEBSTER experimentally.

The second part deals with non-cubic structures in saturation states. Third-order terms are found to be sufficient to explain the magnetic phenomena in this case. In the cubic case, these terms involve no variable part of the energy, and hence the fifth-

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* MAXWELL: 'Electricity and Magnetism,' vol. 2, p. 81.

† [*Cf.* BORN'S method of explaining the phenomenon of optical activity in cubic crystals.]

order terms have to be retained. Formulæ for pyrrhotite are worked out and the same results are obtained as those of Weiss.

The third part attempts to extend the theory to non-saturation states.

Of the three appendices, Appendix B is important. Two interesting series which occur in the theory are numerically calculated. The method was suggested by Mr. A. E. Ingham. Appendix C consists of a few notes.

I wish to thank Prof. C. G. Darwin, F.R.S., who set me on to work at the problem of Webster's experiments, Mr. Webster himself and Dr. Kapitza for occasional discussions with them, Mr. A. E. Ingham and especially Mr. E. Cunningham, St. John's College, for his interest and constant help throughout the course of this work.

PART I.

Chapter 1.

Cubic Structures.

[1] *Preliminary remarks and notation.*—Consider a crystal C. The ultimate particles (or say the atoms) which form the lattice-points of the structure are, so far as magnetic phenomena are concerned, equivalent each to a certain magnetic element. To determine the nature of these elementary magnets is the first important question, but to this we shall return later. The second question is, " how are these magnetic elements disposed in any steady state of the crystal ? " The regularity of the structure suggests the following assumption. In any steady state, half the number of the magnetic elements are pointing in one direction, with the remaining half in some other direction. In the saturation case the two directions coincide, while in the neutral (*i.e.*, non-magnetized) state, they are just opposite.

This assumption of associating *two* directions with any state of magnetization lends itself very easily for application to the principal types we are going to consider, viz., iron and pyrrhotite. The structure of iron is " space-centred cubic," that is to say it consists of two simple cubic lattices interlocked in such a way that the lattice-points of either are the centres of the cells of the other. Likewise, de Waard suggests for pyrrhotite, a hexagonal structure, consisting of two orthorhombic lattices interlocked, as in the case of iron, in the " space-centred-way " We now suppose that in either of these two cases the magnetic elements of the two component simple lattices determine the two directions spoken of in our fundamental assumption. In other words, all the elements of the one lattice point in one direction, while all those of the other, in some other direction.

Let ϑ = Saturation value of magnetization I, for the crystal C.
ψ = Angle between the two directions.

Then obviously, the resultant magnetization of the crystal is

$$\mathrm{I} = 2 \cdot \frac{\vartheta}{2} \cos \tfrac{1}{2}\psi = \vartheta \cos \psi.$$

The following notation will be adopted throughout this paper :—

L_1, L_2 denote the two component simple lattices ;

$(l_1\, m_1\, n_1)$, $(l_2\, m_2\, n_2)$ denote the directions of the magnetic elements in L_1, L_2 respectively, the frame of reference being taken with origin at a lattice-point, and axes along the principal directions of the structure ;

$(l'\, m'\, n')$ denotes the direction of the external field H ;

ω^3 denotes the volume of an elementary cell of either L_1 or L_2 ; so that

ω^{-3} denotes the density of magnetic elements per unit volume in either lattice.

It may be stated expressly that by the term "magnetic element," elementary magnet, or molecular magnet, we simply mean an equivalent simple element which would produce the same magnetic effects as the actual atom with its complex nature does. It is in this sense that we shall speak of the crystal as formed out of magnetic elements building up the lattice-structure.

[2] *Energy of the system.*—Let the crystal C be placed in an external magnetic field H. The effect will be to disturb the original state of its magnetization. To determine the resulting state we apply the general principle of minimum energy. The energy-function cannot be determined without knowing the nature of the magnetic elements. The following alternative conceptions suggest themselves. We may identify the "elements" with—

(i) simple doublets, moment μ ;

(ii) magnets with small yet finite length a, moment μ ;

(iii) elementary magnetic shells, strength μ, effective mean radius a ;

(iv) permanent electronic orbits, with effective moment $= i\,\pi\, a^2 = \mu$;

(v) Current circuits subject to induction.

Now (ii) includes (i) as a particular case ; (iii) and (iv) lead exactly to one and the same form for the energy ; and (v) has the same expression as (iii) and (iv) but with a different sign.

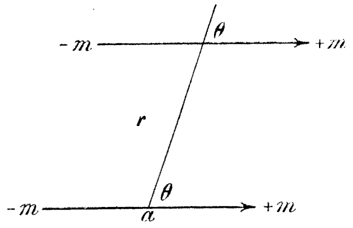
In calculating the energy we shall for the present confine ourselves to the saturation case only, for the following reasons :—

(1) The experimental results in WEBSTER's paper refer mostly to high fields above the saturation region.

(2) In high fields effects of changes in temperature may be neglected.

(2) Though it is conceivable to have alternatives to the fundamental assumption of magnetic elements arranging themselves into two groups with two directions, still there can be no doubt that in the saturation case they all point in the same direction.

Thus the saturation case has the double advantage of involving few theoretical difficulties, and furnishing reliable experimental evidence.



Starting with (ii), *i.e.*, magnets with small measurable length a (bar-magnets),
Let $2\omega_1$ = mutual potential energy of two parallel magnets. Then

$$2\omega_1 = 2m^2/r - m^2\left[(r^2 + a^2 - 2ar\cos\theta)^{-\frac{1}{2}} + (r^2 + a^2 + 2ar\cos\theta)^{-\frac{1}{2}}\right]$$

$$= -2m^2a^2\mathrm{P}_2(\cos\theta)/r^3 - 2\mu^2a^2\mathrm{P}_4(\cos\theta)/r^5 - \ldots,$$

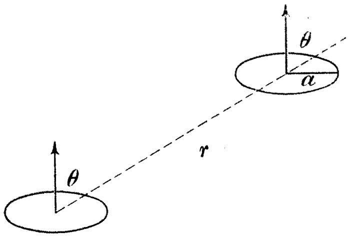
Of this one half is the contribution of the one magnet, and one half of the other.

If we therefore consider the contribution to the energy arising from a magnet, taken at the origin say, due to the presence of the others at the lattice-points, we get

$$\mathrm{W}_1 = \Sigma'\,\omega_1 = -\Sigma'\,(\mu^2\mathrm{P}_2r^{-3} + \mu^2a^2\mathrm{P}_4r^{-5} + \ldots)$$

Let us now consider, as in (iii) or (iv), permanent electronic orbits, circular, with radius a, and axes parallel.

We get the mutual potential energy in the form*—



$$2\,\omega_2 = -2\pi^2i^2a^4\mathrm{P}_2(\cos\theta)\,r^{-3} + 6\pi^2i^2a^6\mathrm{P}_4(\cos\theta)\,r^{-5}\ldots$$

Thus each electronic orbit would be responsible for energy,

$$\omega_2 = -\mu^2\mathrm{P}_2r^{-3} + 3\mu^2a^2\mathrm{P}_4r^{-5}\ldots$$

in the presence of one other.

And hence

$$\mathrm{W}_2 = \Sigma'\,\omega_2 = -\Sigma'\,(\mu^2\mathrm{P}_2r^{-3} - 3\mu^2a^2\mathrm{P}_4r^{-5}\ldots).$$

In the last case (v), we get the same expression, except that the sign is reversed.

$$\mathrm{W}_3 = \Sigma'\,\omega_3 = +\Sigma'\,(\mu^2\mathrm{P}_2r^{-3} - 3\mu^2a^2\mathrm{P}_4r^{-5}\ldots)$$

Now, we have three different forms for the energy according to the assumption we make about the nature of the elementary magnets. It will be seen, when we come to consider WEBSTER's experimental results, that—

(1) They necessitate the presence of terms of the fifth order in $1/r$. This rules out the simple-doublet conception; and

(2) They require positive sign for these terms. This means that we must adopt the W_2 form, *i.e.*, we must suppose that the magnetic properties of the particles arise from the presence of permanent electronic orbits not subject to induction.

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* Appendix A.

[3] *Energy evaluated in the case of cubic structures.*—The "space-centred" cubic structure may be specified analytically as follows :—

Let $\delta = 2\,s =$ lattice-constant,

i.e., the side of an elementary cubic cell of either lattice.

The lattice-points are then represented by

$$x = ps,\ \ y = qs,\ \ z = rs,\ \ [p = q = r \ (\text{mod } 2)].$$

The even values of (pqr) define one of the component lattices, the odd values represent the other.

If now $(lmn) =$ direction in which all the elements are pointing and we consider two magnets, one at the origin, and the other at $(p\,s,\ q\,s,\ r\,s)$ then we have

$$\rho^2 = s^2\,(p^2 + q^2 + r^2) \text{ and } \cos\,\theta = (lp + mq + nr)\,/\,(p^2 + q^2 + r^2)^{\frac{1}{2}}.$$

The expression for internal energy is got by substituting these in the formula

$$W_2 = -\,\Sigma'\,(\mu^2 P_2\,(\cos\,\theta)\,\rho^{-3} - 3\,\mu^2 a^2 P_4\,(\cos\,\theta)\,\rho^{-5}).$$

neglecting terms of the higher order.

We shall show later* that the third-order terms $\Sigma\ \mu^2 P_2/\rho^3$ fall out in the cubic cases. As they stand, they form a divergent series. Strictly speaking our crystals are finite in dimensions, and these terms have to be replaced by a mathematical device—a surface integral. As will be shown later (p. 88), this will not involve any *variable* part of the energy. Anticipating this result, we leave out these terms on nearly the same principle on which infinite constants are omitted in the theory of Attractions and Potentials.

Thus the *effective* internal energy due to one magnet in a space-centred structure is

$$W_2{}' = W = +\,\Sigma'\,3\,\mu^2 a^2 P_4\,(\cos\,\theta)/\rho^5.$$

Now,

$$P_4 = \tfrac{1}{8}\,(35\,\cos^4\,\theta - 30\,\cos^2\,\theta + 3),$$

$$\rho^2 = s^2\,(p^2 + q^2 + r^2),$$

and

$$\cos\,\theta = (lp + mq + nr)/(p^2 + q^2 + r^2)^{1/2}.$$

Hence, the expression for W becomes :

$$\frac{3\,\mu^2 a^2}{8 s^5}\sum_{p,\,q,\,r}\left[\frac{35\,(lp + mq + nr)^4}{(p^2 + q^2 + r^2)^{9/2}} - \frac{30\,(lp + mq + nr)^2}{(p^2 + q^2 + r^2)^{7/2}} + \frac{3}{(p^2 + q^2 + r^2)^{5/2}}\right].$$

$$p = q = r \quad (\text{mod } 2).$$

All the series involved are of terms of the order r^{-5}, and hence are convergent. We may, therefore, although the crystal is finite in dimensions, let p, q, r run through all

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* See § 15.

the integer values from $-\infty$ to $+\infty$ without involving serious error. Further, on account of cubic symmetry, series involving odd powers or products of p, q, r vanish. Thus—

$$\sum_{pqr} \frac{p^3 q}{(p^2+q^2+r^2)^{9/2}} = \sum_{pqr} \frac{p^2 qr}{(p^2+q^2+r^2)^{9/2}} = \sum_{pqr} \frac{pq}{(p^2+q^2+r^2)^{7/2}} = 0, \text{ etc., etc.}$$

And also we have

$$\sum_{pqr} \frac{p^4}{(p^2+q^2+r^2)^{9/2}} = \sum_{pqr} \frac{q^4}{(p^2+q^2+r^2)^{9/2}} = \sum_{pqr} \frac{r^4}{(p^2+q^2+r^2)^{9/2}} = \text{P say,}$$

and similarly,

$$\sum_{pqr} \frac{q^2 r^2}{(p^2+q^2+r^2)^{9/2}} = \sum_{pqr} \frac{r^2 p^2}{(p^2+q^2+r^2)^{9/2}} = \sum_{pqr} \frac{p^2 q^2}{(p^2+q^2+r^2)^{9/2}} = \text{Q, say.}$$

Making use of these facts we see that the variable part of W, *i.e.*, the part depending on the direction $(l\ m\ n)$ of magnetization is

$$105\,\mu^2 a^2\,[\mathrm{P}\,(l^4+m^4+n^4)+6\mathrm{Q}\,(l^2m^2+m^2n^2+n^2l^2)]/\,8s^5.$$

Replacing Σl^4 by $1-2\,\Sigma\,m^2n^2$ we simply get, omitting the constant part,

$$105\,(3\mathrm{Q}-\mathrm{P})\,(l^2m^2+m^2n^2+l^2n^2)/4s^5.$$

This, it will be remembered, is the contribution from one magnetic element. The density of these elements in the crystal is $2\,\omega^{-3}$ per unit volume.

Therefore the internal energy per unit volume is

$$\mathrm{E}_i = 105\,\mu^2 a^2\,(3\mathrm{Q}-\mathrm{P})\,\Sigma\,m^2n^2/2\,\omega^3 s^5.$$

As regards the part arising from the presence of the external field H, it is seen to be

$$\mathrm{E}_{\text{ext}} = -\,2\,\mu\mathrm{H}\,(ll'+mm'+nn')/\,\omega^3.$$

Thus, the total energy density of the crystal C of space-centred cubic structure in an external field H is

$$\mathrm{E} = 2\,\omega^{-3}\,[105\,\mu^2a^2\,(3\mathrm{Q}-\mathrm{P})\,\Sigma\,l^2m^2/4s^5-\mu\mathrm{H}\,\Sigma\,ll'] \;.\;.\;.\;.\;.\;. \quad (1.1)$$

The calculation of the constants P, Q involves great labour (see Appendix B). The results may be quoted here.

$$0{\cdot}16411 < \mathrm{P} < 0{\cdot}17596,$$
$$0{\cdot}06980 < \mathrm{Q} < 0{\cdot}07573,$$
$$0{\cdot}03346 < (3\mathrm{Q}-\mathrm{P}) < 0{\cdot}06307.$$

It should be noted for the present that $3\,\mathrm{Q}-\mathrm{P}$ is essentially *positive* and of the order 10^{-2}.

In what follows various particular cases will be considered, the method being that of relating $(l\ m\ n)$ with $(l'\ m'\ n')$ in such a way as to make the energy-density stationary.

[4] *A circular plate cut parallel to the face of the elementary cube: Webster's experiments.*—A thick circular plate may be regarded as a flattened spheroid—(Maxwell uses the term "planetoid")—and hence as having a bounding surface of the second degree. By a general theorem,* therefore, in an external *homogeneous* field, it will be *homogeneously* magnetized.

Let H act in the plane of the disc, so that $n' = 0$. In virtue of the structural symmetry the resulting magnetization must be also in the plane of the disc. So we have $n = 0$.

Hence the expression for the energy-density (1.1) reduces to:

$$E = 2\omega^{-3}\left[105\mu^2 a^2(3Q - P)\,l^2m^2/4s^5 - \mu H(ll' + mm')\right] \quad . \quad . \quad . \quad . \quad (1.2)$$

To make this stationary, subject to $l^2 + m^2 = 1$, we put

$$\alpha = 105a^2(3Q - P)/4s^5$$

and use T for the essentially variable part of E, so that

$$T = \alpha l^2 m^2 - H(ll' + mm')/\mu.$$

The usual Lagrangian method gives

$$\frac{\partial T}{\partial l}\,dl + \frac{\partial T}{\partial m}\,dm = 0, \qquad l\,dl + m\,dm = 0,$$

i.e.,

$$\begin{vmatrix} \dfrac{\partial T}{\partial l} & \dfrac{\partial T}{\partial m} \\ l & m \end{vmatrix} = 0.$$

This reduces to

$$\frac{\begin{vmatrix} l & m \\ l' & m' \end{vmatrix}}{lm(l^2 - m^2)} = \frac{2\alpha\mu}{H}.$$

To put this result in a neat form, let OX, OY be the principal axes of the structure in the plane of the disc, and let ϕ, ψ be the angles made by H and I respectively with OX. Then

$$(l', m') = (\cos\phi, \sin\phi),$$
$$(l, m) = (\cos\psi, \sin\psi).$$

And the formula reduces to

$$\frac{H\sin(\phi - \psi)}{\mu\sin\psi\cos\psi\cos 2\psi} = 2\alpha, \text{ (a positive constant)}, \textit{ i.e.}, \frac{H\sin(\phi-\psi)}{\mu\sin 4\psi} = \tfrac{1}{2}\alpha. \quad (1.3)$$

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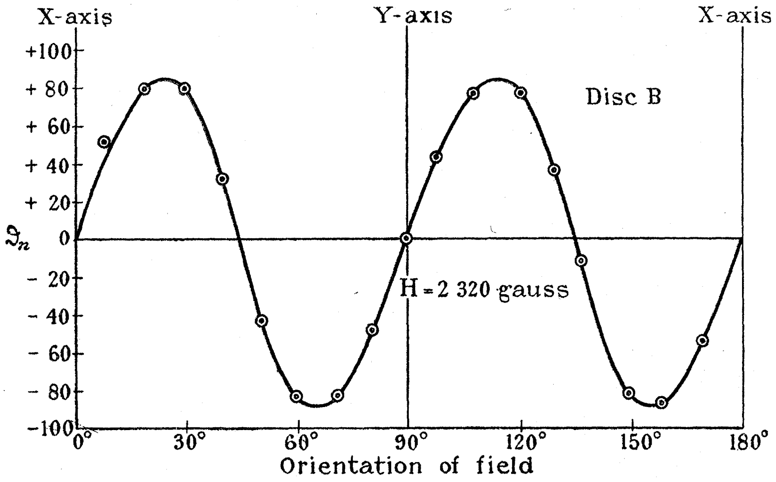
* Maxwell: 'Electricity and Magnetism,' vol. 2, p. 67.

This result accounts for WEBSTER's results completely :—

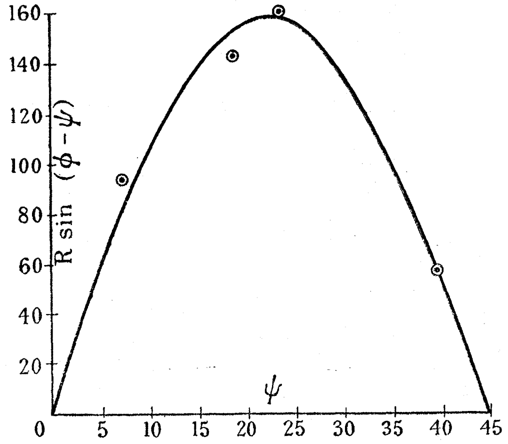
(1) In the first place we see that the deviation $\delta = |\phi - \psi|$ vanishes when

$$\sin 4\psi = 0,$$

i.e., the direction of the resultant magnetization of the plate coincides with that of the external field along the principal axes of the structure, or exactly mid-way between them only. This is borne out by the following graph* :—



(2) Secondly, we see that maximum deviation occurs when $\sin 4\psi$ is maximum, *i.e.*, when $\psi = \pi/8$. This fits in with WEBSTER's graph 5 (*loc. cit.*, p. 506), reproduced below :—



(3) Thirdly, we get from the above

$$H \sin \delta_{\text{max.}} = \tfrac{1}{2}\alpha\mu. \qquad \text{(1.4)}$$

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* WEBSTER : "The Magnetic Properties of Iron Crystals," Graph 2, 'Roy. Soc. Proc.,' A., vol. 107, p. 503 (1925).

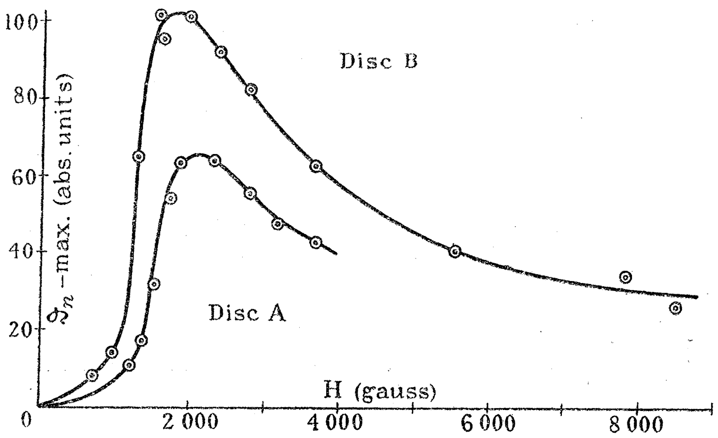
WEBSTER'S experiments refer to fields above saturation region. Multiplying both sides by ϑ

$$H \times \vartheta \sin \delta_{\text{max.}} = \tfrac{1}{2}\alpha\,\mu\vartheta, \quad (= \text{constant}).$$

Now, $\vartheta \sin \delta$ is the component of magnetization perpendicular to H, ϑ_n say. Hence we have

$$H \,.\, (\vartheta_n)_{\text{max.}} = \tfrac{1}{2}\alpha\,\mu\vartheta. \quad \ldots\ldots\ldots\ldots \quad (1.5)$$

Equation (1.5) means that above the saturation region the graph of H and ϑ_n is a rectangular hyperbola. This is supported by WEBSTER'S graph 4 (*loc. cit.*, p. 505) :—



In this case the experiments of KARL BECK also bear out the present result :—

> " Die Maximums [of ϑ_n] in den starken Fieldern ist, wie die Angaben über das Drehmoment zeigen werden, eine hyperbolische, indem $\vartheta_{\text{max.}} \,.\, H$ konstant wird, oder $\sigma_{\text{max.}}$ proportional zu $1/H$; es würde also für $H = \infty$, $\sigma_n = 0$ für alle Azimute."*

Remarks.—I. Let us compare the result, we obtained above,

$$H \sin (\phi - \psi)/\mu \sin \psi \cos \psi \cos 2\psi = \text{constant}$$

with WEISS'S result,

$$H \sin (\phi - \psi)/I \sin \psi \cos \psi = \text{constant},$$

which he obtains in the case of pyrrhotite. Now pyrrhotite has a non-cubic structure, and when we come to deal with non-cubic structures, exactly the same result will be arrived at. It need only be observed here that the presence of the extra factor $\cos 2\psi$ in the denominator corresponds to the experimental fact that the deviation-effect vanishes when H is applied at 45° to either of the principal axes.

II. The second observation that may be made is important. By adopting the ω_2 form for the energy, we obtained the present result,

$$H \sin (\phi - \psi)/\mu \sin \psi \cos \psi \cos 2\psi = \text{a } \textit{positive} \text{ constant.}$$

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* KARL BECK, " Das magnetische Verhalten von Eisenkristallen bei gewöhnlicher Temperatur," p. 133.

This means that

$$\phi - \psi > 0 \quad \text{if} \quad \psi < \tfrac{1}{4}\pi$$

and

$$\phi - \psi < 0 \quad \text{if} \quad \psi > \tfrac{1}{4}\pi.$$

In other words the magnetization I lies somewhere between the external field H and that principal axis which is nearer to H. This is experimentally true, although WEBSTER does not state it explicitly.

If we had adopted any of the other forms for the energy, which involved the fifth-order terms with a negative sign, an exactly opposite result would have been arrived at. Hence, it is that we adopted the form

$$\omega_2 = - \mu^2 P_2 r^{-3} + 3\mu^2 a^2 P_4 r^{-5}.$$

So far we have applied only qualitative tests. Quantitative tests are postponed to the third chapter.

[5] *Plate cut parallel to the face of an octahedron.*—We start with the experimental fact :—" The amplitude of the waves of ϑ_n reaches a small maximum in small fields and vanishes in stronger fields almost completely."* It will be shown now that the present theory bears out this result.

The eight faces of an octahedron formed from a cubic structure have for their normals $(\pm 1/\sqrt{3},\ \pm 1/\sqrt{3},\ \pm 1/\sqrt{3})$ as direction cosines. Consider, in particular, the face whose normal is $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. Clearly if (λ, μ, ν) be any vector in the plane of the disc

$$\lambda + \mu + \nu = 0.$$

Now, H is applied in the plane of the disc, and in virtue of the symmetry I also lies in the plane. Hence

$$l' + m' + n' = 0, \qquad l + m + n = 0.$$

Subject to the conditions,

$$\Sigma\, l = 0, \qquad \Sigma\, l^2 = 1,$$

we have to make stationary the expression

$$T = \alpha\,(l^2m^2 + m^2n^2 + n^2l^2) - H\,(ll' + mm' + mn')/\mu.$$

The usual procedure gives us :

$$\begin{vmatrix} \dfrac{\partial T}{\partial l} & \dfrac{\partial T}{\partial m} & \dfrac{\partial T}{\partial n} \\ l, & m, & n \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

i.e.,

$$-2\alpha \begin{vmatrix} l^3 & m^3 & n^3 \\ l & m & n \\ 1 & 1 & 1 \end{vmatrix} - \frac{H}{\mu} \begin{vmatrix} l' & m' & n' \\ l & m & n \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

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* KUNZ, 'Bull. Nat. Res. Council,' vol. 3, p. 183 (1922).

Now, the coefficient of α contains $(l + m + n)$ as a factor, and hence vanishes. Therefore

$$\begin{vmatrix} l' & m' & n' \\ l & m & n \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

which taken with $\Sigma l = \Sigma l' = 0$ leads to

$$l/l' = m/m' = n/n',$$

i.e., H and I are in the same direction, *i.e.*, ϑ_n vanishes completely.

Thus the theory, so far as it is worked out (saturation cases), confirms the experimental result as quoted from KUNZ's paper.

[6] *A plate cut parallel to the face of the rhombic dodecahedron.*—A rhombic dodecahedron is a semi-regular solid; and a cubic structure, such as we are now concerned with, can be partitioned into identical cells of this shape without leaving any gaps. The simplest way to describe its construction is perhaps the following one: Take two equal cubical blocks A and B. Divide B into six equal pyramids having the centre of B as the common vertex and the six faces, as their bases. Take these six pyramids and fix them on to the faces of the cube A. The resulting solid will be a rhombic dodecahedron. It has twelve faces, fourteen vertices and twenty-four edges. All the faces are identical and rhombic, but not rectangular and hence not regular.

In terms of our space-centred cubic structure, if we take for A, a cell of the L_1-lattice, the vertices of the pyramids, which are joined on to the faces of A, will occupy the centres of the six cells of L_1, which immediately surround it. Thus, these six out of the total number of fourteen vertices, are the lattice-points of the L_2-lattice.

It is easy to see that the twelve faces have for their normals the direction-cosines:

$$(0, \pm 1/\sqrt{2}, \pm 1/\sqrt{2})\,; \quad (\pm 1/\sqrt{2}, 0, \pm 1/\sqrt{2})\,; \quad (\pm 1/\sqrt{2}, \pm 1/\sqrt{2}, 0).$$

Consider a plate cut parallel to the face $(1/\sqrt{2}, 1/\sqrt{2}, 0)$.

The experimental result as given by KUNZ is:—"The plate contains at 0° and 180° a quarternary axis, at 90° one binary axis, and 55° and 125° a ternary axis … . The normal components ϑ_n show again four zeros within 180° in the direction of the axes at 0°, 55°, 125° and 180°."*

We proceed to show that this result is confirmed by the present theory.

H is applied in the plane $(1/\sqrt{2}, 1/\sqrt{2}, 0)$, and since it is a symmetry plane, I also lies in the plane. Hence

$$l' + m' = 0, \qquad l + m = 0.$$

To make stationary, the expression

$$\mathrm{T} = \alpha\, \Sigma\, l^2 m^2 - \mathrm{H}\, \Sigma\, l l'/\mu$$

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* KUNZ, 'Bull. Nat. Res. Council,' vol. 3, Part 3, p. 182–3 (1922).

subject to $\Sigma l^2 = 1$, $l + m = 0$, we get, as usual

$$\begin{vmatrix} \frac{\partial T}{\partial l} & \frac{\partial T}{\partial m} & \frac{\partial T}{\partial n} \\ l & m & n \\ 1 & 1 & 0 \end{vmatrix} = 0.$$

i.e.,

$$2\alpha \begin{vmatrix} l^3 & m^3 & n^3 \\ l & m & n \\ 1 & 1 & 0 \end{vmatrix} + \frac{H}{\mu} \begin{vmatrix} l' & m' & n' \\ l & m & n \\ 1 & 1 & 0 \end{vmatrix} = 0.$$

Obviously, whenever

$$\begin{vmatrix} l^3 & m^3 & n^3 \\ l & m & n \\ 1 & 1 & 0 \end{vmatrix} = 0$$

we must have

$$\begin{vmatrix} l' & m' & n' \\ l & m & n \\ 1 & 1 & 0 \end{vmatrix} = 0,$$

which coupled with $l + m = l' + m' = 0$ means $(lmn) = (l'm'n')$

Thus, the zeros of I_n are given by

$$\begin{vmatrix} l^3 & m^3 & n^3 \\ l & m & n \\ 1 & 1 & 0 \end{vmatrix} = 0$$

with, of course,

$$l^2 + m^2 + n^2 = 1, \qquad l + m = 0.$$

Putting $l = - m$, in the above we get

$$\begin{vmatrix} -m^3 & m^3 & n^3 \\ -m & m & n \\ 1 & 1 & 0 \end{vmatrix} = 0,$$

i.e., $mn\,(m^2 - n^2) = 0$.

Hence,

(i) $m = 0$, when, of course, $l = 0$ and $n = \pm 1$ so that we get a quaternary axis.

(ii) $n = 0$, and then $l = - m = \pm 1/\sqrt{2}$ and we get a binary axis.

and lastly (iii) $m^2 - n^2 = 0$ and since $l = - m$, we get $l^2 = m^2 = n^2 = 1/3$.

This gives *two distinct* directions, and they make with the quaternary axis in case (i) angles,

$$\beta_1, \beta_2 = \cos^{-1} (0 + 0 \pm 1/\sqrt{3})$$
$$= \cos^{-1} 1/\sqrt{3}, \quad \pi - \cos^{-1} 1/\sqrt{3}.$$

Now $\cos^{-1}(1/\sqrt{3}) = 54° \, 44' = 55°$ approximately.

Therefore the two directions make angles 55° and 125° with the quaternary axis.

Thus we see that the theory completely accounts for the four zeros of ϑ_n at 0°, 55°, 90°, 125°, and then at 180°, within the range of 180°.

CHAPTER II.

The Molecular Field.

[7] *The conception of* WEISS'S *molecular field.*—The " molecular field " was first introduced in connection with pyrrhotite in the following way.

It is supposed, in the first place, that the magnetization I of a body is the effect of the joint influence of the external field H, and the internal field H_m which latter WEISS calls the molecular field. Secondly, it is assumed that the components of H_m along the structural axes of the body are proportional to the components of I in those directions. The proportionality factors need not be the same for the three axes.

Symbolically, we have

$$H_{\text{total}} = H_{\text{applied}} + H_m \quad \text{and} \quad H_{mx} = N_1 I_x, \quad H_{my} = N_2 I_y, \quad H_{mz} = N_3 I_z.$$

From the view-point of theory, the molecular field is only a bulk-conception in the sense that it is only (or at least meant to be) an effective substitute for the microscopic actions, and actions of other nature possibly, which are involved, besides the applied H, in determining the magnetization of the body. It has the same theoretical standing as the elastic force D in WEBER'S Theory of induced magnetism.

Let us apply these ideas to cubic structures. None of the principal axes of such a structure could be distinguished from the other two, and hence we must put

$$N_1 = N_2 = N_3 = N, \text{ say.}$$

This would mean that H_m and I will be always in the same direction. Hence if, further, the boundary of the body be symmetrical, H, H_m and I will be in the same direction. In other words no deviation-effect would exist. But actually, as WEBSTER'S experiments show, this conclusion is not borne out by experimental evidence. Now, WEBSTER suggests that WEISS'S hypothesis of simple proportionality between H_m and I should be given up, in favour of his own, viz.,

$$(H_m)_x = I \cos^4 \psi,$$
$$(H_m)_y = I \sin^4 \psi.$$

To quote WEBSTER fully : " Since iron has a cubic structure, one would expect the constant N to be the same for both axes. WEISS found the function $f(\psi)$ to be simply $\cos \psi$. In our case this cannot be, for the resultant molecular field would have no component perpendicular to the direction of magnetization. The value of $f(\psi) = \cos^4 \psi$ was found to give the best agreement with experimental evidence."*

To examine this point closely, let us start with WEISS's hypothesis—

$$(H_m)_x = N_1 I_x = N_1 I l,$$
$$(H_m)_y = M_2 I_y = N_2 I m.$$

The rational way to modify this, is to generalize it into

$$\left.\begin{aligned}(H_m)_x &= N_1 f(I_x) = N_1 f(Il) \\ (H_m)_y &= N_2 f(I_y) = N_2 f(Im)\end{aligned}\right\}.$$

In the saturation case $I = \vartheta$, and we might put

$$(H_m)_x = N_1 \vartheta f(l), \quad (H_m)_y = N_2 \vartheta f(m).$$

For the sake of generality we adopt the first form. The expression for the energy of the system in an external field H will be $-(H_{\text{total}}\,.\,I)$, that is

$$-(HI) - \int_0^I (H_m\, dI) = -|H|\ |I|\,(ll' + mm') - \int_0^I [N_1 f(lI)\, l + N_2 f(mI)\,.\,m]\, dI.$$

Assume now that the function f is of the form :—

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots .$$

Obviously $a_0 = 0$, since there is to be no molecular field in the non-magnetized state. The expression for the energy becomes

$$-|H|\,.\,|I|\,(ll' + mm') - [\{\tfrac{1}{2}a_1 I^2 l^2 + \tfrac{1}{3}a_2 I^3 l^3 + \dots\}\, N_1 + \{\tfrac{1}{2}a_1 I^2 m^2 + \tfrac{1}{3}a_2 I^3 m^3 + \dots\}\, N_2].$$

The steady states are obtained by making the above expression, say T, stationary, subject to $(l^2 + m^2) = 1$.

We get as usual

$$\begin{vmatrix} \partial T/\partial l & \partial T/\partial m \\ l & m \end{vmatrix} = 0.$$

This reduces to

$$\frac{H \begin{vmatrix} l & m \\ l' & m' \end{vmatrix}}{Ilm\,[a_1 (N_1 - N_2) + a_2 (lN_1 - mN_2)\, I + \dots + a_r (l^{r-1} N_1 - m^{r-1} N_2)\, I^{r-1} + \dots]} = 1.$$

This is the most general relation that we can have. In WEISS's hypothesis,

$$f(lI) = lI, \text{ simply,}$$

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* "Magnetic Properties of Iron Crystals," by W. L. WEBSTER, 'Roy. Soc. Proc.,' A, vol. 107, p. 506 (1925).

i.e., $a_1 = 1$, and $a_2 = a_3 = - 0$, and we get

$$\mathrm{H}\begin{vmatrix} l & m \\ l' & m' \end{vmatrix} \Big/ \mathrm{I}lm\,(\mathrm{N}_1 - \mathrm{N}_2) = 1,$$

or, using the familiar notation,

$$\mathrm{H}\sin(\phi - \psi)/\mathrm{I}\sin\psi\cos\psi = (\mathrm{N}_1 - \mathrm{N}_2), \text{ constant}.$$

Now to apply the general formula to the cubic case we put $\mathrm{N}_1 = \mathrm{N}_2$, and as a next approximation retain only the first non-vanishing term in the denominator.

We get, if $a_2 \neq 0$.

$$\mathrm{H}\begin{vmatrix} l & m \\ l' & m' \end{vmatrix} \Big/ \mathrm{I}^2 lm\,(l - m)\,\mathrm{N} = 1.$$

This result, however, does not satisfy all the requirements of cubic symmetry. For the experimental result is that $(l, m) = (l', m')$ along the principal axes and also exactly midway between them. In other words I is in the same direction as H, when H acts in a direction inclined to Ox at any of the following angles

$$(0^\circ, \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2, 7\pi/4),$$

i.e., when

$$lm\,(l^2 - m^2) = 0.$$

Hence the result we obtained above, which does not contain a factor $(l + m)$ must be rejected. It is easy to see that to have the factor $(l^2 - m^2)$, all the non-vanishing terms must have an odd suffix for a. Hence we see that

$$a_2 = a_4 = a_6 = \ldots = 0.$$

Thus the formula in the cubic case becomes

$$\mathrm{H}\begin{vmatrix} l & m \\ l' & m' \end{vmatrix} \Big/ \mathrm{I}lm\,[\mathrm{I}^2(l^2 - m^2)\,a_3 + \mathrm{I}^4(l^4 - m^4)\,a_5 + \ldots] = \mathrm{N}.$$

Retaining only the first term, we have

$$\mathrm{H}\begin{vmatrix} l & m \\ l' & m' \end{vmatrix} \Big/ \mathrm{I}lm\,(l^2 - m^2) = a_3\mathrm{N}\mathrm{I}^2.$$

And since we have been dealing with saturation cases, $\mathrm{I} = \vartheta$, and we write the above result in the form

$$\mathrm{H}\sin(\phi - \psi)/\vartheta\sin\psi\cos\psi\cos 2\psi = a_3\mathrm{N}\,.\,\vartheta^2,$$

which is the same result as obtained before (1.3).

Remarks.—I. It should be noted that the fact that $a_{2r} = 0$ for all r at once rules out an even-power law for the molecular field. We further see that a relation

$$\left.\begin{aligned} (\mathrm{H}_m)_x &= \mathrm{N}a_3\mathrm{I}^3l^3 \\ (\mathrm{H}_m)_y &= \mathrm{N}a_3\mathrm{I}^3m^3 \end{aligned}\right\}$$

yields the same result as was arrived at by the atomistic theory. The same conclusion could be seen from another view-point.

II. We have seen before that on the conception of mutual actions of the magnetic elements the effective total energy in its modified form was

$$\mathrm{T} = \alpha l^2m^2 - \frac{\mathrm{H}}{\mu}(ll' + mm').$$

If we now introduce the bulk-conception of the molecular field with a third-power law as stated above, we shall get an exactly similar expression for T.

$$\begin{aligned} \text{Thus, energy} &= -\int_0^{\mathrm{I}} (\mathrm{H}_{\text{total}}\, d\mathrm{I}) = -\int_0^{\mathrm{I}} (\mathrm{H}_m\, d\mathrm{I}) - (\mathrm{HI}) \\ &= -\int_0^{\mathrm{I}} a_3\mathrm{N}\,(l^3\mathrm{I}^3\,.\,l d\mathrm{I} + m^3\mathrm{I}^3\,.\,m\,d\mathrm{I}) - \mathrm{HI}\,(ll' + mm') \\ &= -\tfrac{1}{4}a_3\mathrm{NI}^4(l^4 + m^4) - \mathrm{HI}\,(ll' + mm') \end{aligned}$$

In the saturation case, $\mathrm{I} = \vartheta$, a constant, and $l^4 + m^4 = 1 - 2l^2m^2$. Hence, omitting the constant part, the effective energy depending on (l, m) is

$$\tfrac{1}{2}a_3\mathrm{N}\vartheta^4\,.\,l^2m^2 - \mathrm{H}\vartheta\,(ll' + mm'),$$

which is exactly of the form $\alpha l^2m^2 - \frac{\mathrm{H}}{\mu}(ll' + mm')$ obtained before.

III. The point of generalizing Weiss's hypothesis in the way we have done is that we need not have two different laws for the molecular field, the one for the cubic case, and the other for the non-cubic case. One single relation :

$$\left\{\begin{aligned} (\mathrm{H}_m)_x &= \mathrm{N}_1\,[a_1l\mathrm{I} + a_3l^3\mathrm{I}^3 + a_5l^5\mathrm{I}^5 + \ldots] \\ (\mathrm{H}_m)_y &= \mathrm{N}_2\,[a_1m\mathrm{I} + a_3m^3\mathrm{I}^3 + \ldots]. \end{aligned}\right.$$

covers both cases. For the result we get by making the energy-function stationary is

$$\mathrm{H}\begin{vmatrix} l & m \\ l' & m' \end{vmatrix} \Big/ \mathrm{I}lm\left[\sum_0^\infty a_{2r+1}\,(l^{2r}\,\mathrm{N}_1 - m^{2r}\,\mathrm{N}_2)\,\mathrm{I}^{2r}\right] = 1.$$

In the non-cubic case $\mathrm{N}_1 \neq \mathrm{N}_2$, and retaining only the first term we get Weiss's result. In the cubic case $\mathrm{N}_1 = \mathrm{N}_2$, and the term in a_1 disappears automatically. Retaining therefore the next one in a_3 (which practically amounts to starting with simply a third-power law for the molecular field right from the beginning), we get a formula for the cubic case.

[8] *Agreement with* WEBSTER's *curves.*—To complete the discussion, it remains to show that with a third-power law the agreement with experimental curves continues to be just as good as with the $\cos^4\psi$ law of WEBSTER.

The graph (graphs of WEBSTER reproduced above) is drawn by taking

$$\mathrm{R}\sin(\phi-\psi)=\mathrm{M}\,[f(\psi)\sin\psi-f(90^\circ-\psi)\cos\psi]\ \text{with}\ f(\psi)=\cos^4\psi.$$

The curve with $f(\psi)=\cos^3\psi$ is obtained from the above curve itself, by simply altering the ordinates in the ratio r, where

$$r(\psi)=\frac{\cos^3\psi\,.\,\sin\psi-\sin^3\psi\cos\psi}{\cos^4\psi\sin\psi-\sin^4\psi\cos\psi}=\frac{\cos\psi+\sin\psi.}{1+\cos\psi\sin\psi}.$$

It is easy to see how slightly this fraction differs from unity in the range we are concerned with (0, 45°). We have actually

$$r(0)=r(90^\circ)=1\cdot0000 \qquad r(30^\circ)=r(60^\circ)=0\cdot9532$$
$$r(10^\circ)=r(80^\circ)=0\cdot9892 \qquad r(40^\circ)=r(50^\circ)=0\cdot9440$$
$$r(20^\circ)=r(70^\circ)=0\cdot9699 \qquad r(45^\circ)=r(45^\circ)=0\cdot9428$$

Thus $r(\psi)$ differs from unity so slightly that the new curve will be practically the same as given by WEBSTER—only it will be slightly lower than his. Besides, in four of WEBSTER's graphs, not published in his paper, the experimental points do actually lie below his curves drawn according to the $\cos^4\psi$ law. These, therefore, further strengthen the adoption of the third-power relation.

[9] *Additional evidence in support of a third-power relation for the molecular field.*—In his thesis, KARL BECK obtains for the energy due to the molecular field the following expression

$$\Pi_z=\mathrm{A}\,(\sin^2 2\theta+\sin^4\theta\,.\,\sin^2 2\phi),$$

where $(\theta,\ \phi)$ specify the direction of the magnetization, in polar co-ordinates, *i.e.*,

$$\mathrm{I}_x=\mathrm{I}\sin\theta\cos\phi\,;\quad \mathrm{I}_y=\mathrm{I}\sin\theta\sin\phi\,;\quad \mathrm{I}_z=\mathrm{I}\cos\phi.$$

We shall obtain the same result by our present theory, *i.e.*, by adopting the third-power relation :

$$(\mathrm{H}_m)_x=\mathrm{N}a_3\mathrm{I}^3l^3=\mathrm{N}a_3\mathrm{I}^3(\sin\theta\cos\phi)^3$$
$$(\mathrm{H}_m)_y=\mathrm{N}a_3\mathrm{I}^3m^3=\mathrm{N}a_3\mathrm{I}^3(\sin\theta\sin\phi)^3$$
$$(\mathrm{H}_m)_z=\mathrm{N}a_3\mathrm{I}^3n^3=\mathrm{N}a_3\mathrm{I}^3\cos^3\theta.$$

$$\begin{aligned}\Pi_z&=-\int_0^{\mathrm{I}}(\mathrm{H}_m\,d\mathrm{I})=-\int_0^{\mathrm{I}}(\mathrm{H}_{mx}\,d\mathrm{I}_x+\mathrm{H}_{my}\,d\mathrm{I}_y+\mathrm{H}_{mz}\,d\mathrm{I}_z)\\&=-\int_0^{\mathrm{I}}\mathrm{N}a_3\mathrm{I}^3(l^4+m^4+n^4)\,d\mathrm{I}=-\tfrac{1}{4}\mathrm{N}a_3\mathrm{I}^4(l^4+m^4+n^4).\\&=-\tfrac{1}{4}\mathrm{N}a_3\mathrm{I}^4(1-2\,\Sigma\,m^2n^2).\end{aligned}$$

But

$$\Sigma m^2 n^2 = \sin^4\theta\cos^2\phi\sin^2\phi + \cos^2\theta\,(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi)$$
$$= \tfrac{1}{4}\sin^4\theta\sin^2 2\phi + \tfrac{1}{4}\sin^2 2\theta.$$

Hence we get

$$H_z = \tfrac{1}{8}\,Na_3I^4\,(\sin^2 2\theta + \sin^4\theta\sin^2 2\phi) - \tfrac{1}{4}Na_3I^4.$$

In the saturation case I = a constant, ϑ, and we have then

$$H_z = A\,(\sin^2 2\theta + \sin^4\theta\sin^2 2\phi) - 2A.$$

And since for energy variations for different directions of saturation, the constant part $-2A$ may be left out, we arrive at Beck's result.

$$H_z = A\,(\sin^2 2\theta + \sin^4\theta\sin^2 2\phi).$$

Chapter 3.

Quantitative Tests.

[10] *Determination of the order of magnitude of "a," the radius of an electronic orbit.*—We apply the formula (1.5)

$$H\,(\vartheta_n)_{\text{max.}} = \tfrac{1}{2}\alpha\,\mu\vartheta,$$

where $\alpha = 105a^2(3Q - P)/4s^5$ to Webster's graph 4.

That curve has been drawn with the following values*:

$$k = (H\vartheta_n) = \tfrac{1}{2}\alpha\,\mu\vartheta.$$

Crystal B.

H.	$(\vartheta_n)_{\text{max.}}$	k_B.
8310	29·7	246807
7620	34·5	262890
5530	43	237790
3500	64	244000
2030	96·5	195895

Average k_B = 23347.

Crystal A.

H.	$(\vartheta_n)_{\text{max.}}$	k_A.
3640	44·4	161616
3100	49·1	152210
2740	53·3	146042
2150	63·4	136310

Average k_A = 149044.

So strictly speaking the tables above do not give the product $H\,.\,(\vartheta_n)_{\text{max.}}$ constant. In the case of both crystals, it slightly increases with H, except in one case. In the case of B crystal, as the field increases from 7620 to 8310 the product $H\,.\,(\vartheta_n)_{\text{max.}}$ falls from 262890 to 246807. The theoretical curve, which is a rectangular hyperbola will not account for this increase, nor again the sudden drop. But as Webster himself says in his paper, the measurements of high fields is very difficult, and we cannot secure accurate results. What we shall do is to take the average value of the numbers in the third column, and adopt it as the experimental constant for the rectangular hyperbola.

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* Kindly supplied to the writer by Mr. Webster.

Crystal B.

$$\bar{k}_{\mathrm{B}} = 233476.$$

On comparing this with the theoretical constant we find

$$\frac{105}{8s^5} \cdot \mu a^2 \,(3\,\mathrm{Q} - \mathrm{P})\,\vartheta = 233476.$$

We have now the following data :

ϑ = saturation value in iron = 1620 abs. units (WEBSTER).

The structure of iron is space-centred cubic and the lattice-constant d is*

$$d = 2s = 2\cdot 8^{6}_{3} \times 10^{-8}.$$

There are two lattice-points per cell ; hence if N_0 is the number of lattice-points per cubic centimetre,

$$\mathrm{N}_0 = 2/[2\cdot 8^{6}_{3} \times 10^{-8}]^3 = 8\cdot {}^{82}_{54} \times 10^{22}.$$

Also, a WEISS magneton = $18\cdot 5 \times 10^{-22}$.
Therefore we find for μ, the number of magnetons associated with each lattice-point (*i.e.*, the moment of a magnetic element),

$$\begin{aligned}\mu &= \vartheta/\{\mathrm{N}_0 \times (\text{Value of Weiss's magneton})\}\\ &= 1620/(8\cdot 82\,.\,10^{22} \times 18\cdot 5\,.\,10^{-22}),\ \text{or}\ 1620/(8\cdot 54\,.\,10^{22} \times 18\cdot 5\,.\,10^{-22}).\\ &= 1620/163\cdot 17,\ \text{or}\ 1620/157\cdot 99 = 9\cdot 92,\ \text{or}\ 10\cdot 2.\end{aligned}$$

Thus we get

$$\mu = 10 \text{ Weiss magnetons} = 2 \text{ Bohr magnetons}.$$

Next, adopting $d = 2s = 2\cdot 86 \times 10^{-8}$ we find

$$s^5 = 5\cdot 97 \times 10^{-40}.$$

Substituting these values we get

$$233476 = \tfrac{1}{8}105\mu a^2\,(3\mathrm{Q} - \mathrm{P})\,1620/5\cdot 97\,.\,10^{-40},$$

i.e., $\mu a^2 = 65\cdot 5\,.\,10^{-40}/(3\mathrm{Q} - \mathrm{P})$.

Since μ = 10 WEISS magnetons we get

$$a^2 = 0\cdot 35\,.\,10^{-18}/(3\mathrm{Q} - \mathrm{P}).$$

We know from actual calculation† that $0\cdot 032 < (3\mathrm{Q} - \mathrm{P}) < 0\cdot 064$.

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* SOMMERFELD, 'Atomic Structure and Spectral Lines,' p. 537.

† See Appendix B.

Hence

$$0\cdot4 \times 10^{-8} > a > 0\cdot2 \times 10^{-8}. \qquad (3.1)$$

Crystal A.

Here $k_A = 149044$.

In calculating a, we have seen that a^2 varies as k, therefore the limits for a as obtained from crystal A could be easily deduced from those above from B, by simply multiplying by $\sqrt{k_A/k_B}$.

$$\sqrt{\frac{k_A}{k_B}} = \sqrt{\frac{149044}{233476}} = 0\cdot6383.$$

Hence we deduce for the A crystal the result

$$0\cdot212 > a \times 10^{8} > 0\cdot147. \qquad (3.2)$$

Collecting together the results we have

$$\left.\begin{array}{l} 0\cdot3307 \times 10^{-8} > a_B > 0\cdot2338 \times 10^{-8} \\ 0\cdot212 \quad \times 10^{-8} > a_A > 0\cdot147 \quad \times 10^{-8} \end{array}\right\}. \qquad (3.3)$$

These results are unsatisfactory in one sense, viz., we do not get a common range for a_B and a_A. This discrepancy *may* be due to the presence of impurities in the two crystals in unequal proportions. In any case we conclude that the radius of an electronic orbit is

$$a = O\,[10^{-9}] \qquad (3.4)$$

This result is perfectly reasonable. It shows that since the distance between two lattice-points is at least of the order of 10^{-8}, the electronic orbits of the two will not interfere with each other.

[11] *The order of magnitude of the molecular field in iron.*—In the last chapter we obtained for the energy-function two expressions,
from the bulk-theory,

$$\tfrac{1}{2}a_3 N\vartheta^4 l^2 m^2 - H\vartheta\,(ll' + mm'),$$

and from atomistic-theory,

$$\alpha l^2 m^2 - \frac{H}{\mu}(ll' + mm').$$

Comparison shows that the atomic actions are effectively replaced by the molecular field, if we put

$$a_3 N\vartheta^3 = 2\alpha\,\mu. \qquad (3.5)$$

But, as we have seen, according to our third-power law, $a_3N\vartheta^3$ is exactly the value of the molecular field when H acts along a principal axis. It represents, therefore, what WEBSTER denotes by M. Now $\frac{1}{2}\alpha\,\mu\vartheta = \bar{k}_B = 233476$ and $\vartheta = 1620$, so that

$$M_B = 2\alpha\,\mu = 576. \qquad (3.6)$$

For the A disc, we have

$$\bar{k}_A = 149044.$$

Hence

$$M_A = 2\alpha\mu = 368. \qquad \ldots\ldots\ldots\ldots \quad (3.7)$$

The values suggested by WEBSTER are $M_B = 620$, $M_A = 470$.

Thus we find that the present theory gives the molecular field of the right order of magnitude.

To sum up :

$$a = \text{radius of an electronic orbit} = O[10^{-9}];$$

	Present Theory.	WEBSTER'S Experiments.
M_B	576	620
M_A	368	470

PART II.

CHAPTER 4.

The Non-cubic Structure : Saturation.

[12] *The scope of the present part and introductory remarks.*—The principal aim throughout this part is to give methods of dealing with the third-order terms in the energy-function. When we obtained in the last section the expression

$$W_2 = \Sigma'\,\omega_2 = -\Sigma'\,(\mu^2 P_2/r^3 - 3\mu^2 a^2 P_4/r^5)$$

for the energy, we remarked that the third-order terms which form a divergent series have to be dealt with in a special way. Anticipating the result, we assumed throughout the discussion that these terms can be left out in cubic structures with a symmetrical boundary, as the terms which effectively take their place do not involve any variable part of the energy. This result will be demonstrated in this section. But since, as will be shown, they cannot be neglected in the case of non-cubic structures, the problem of pyrrhotite will be also considered.

In his thesis, DE WAARD devotes a section to "Atomistiese Theorie von het Ferromagnetisme." Therein he has adopted the conception of simple doublets, which are responsible for the third-order terms in $1/r$ in the energy-function. His method of dealing with these terms does certainly give a complete, though only formal, solution. Unfortunately the results involve series of terms each of which is a complicated Hankel function, and it is extremely difficult to check results by numerical data. We shall, however, follow his methods and obtain results slightly more general than his, and establish only qualitative agreement with experimental facts.

[13] *The principle of* DE WAARD'S *method.*—For the sake of generality we shall not

assume at first that there is saturation. According to our fundamental assumptions let $(l_1\ m_1\ n_1)$ and $(l_2 \cdot m_2\ n_2)$ denote the directions in which the elements in L_1 and L_2 lattices respectively are pointing, in the presence of an external homogeneous field H. Now, on the macroscopic theory of magnetism, uniform magnetization of a body is secured in presence of a homogeneous field if (i) the body is continuous, and if, further (ii), it is bounded by a surface of the second degree. To avoid the theoretical difficulties, therefore, let us suppose that our crystal C is in the form of an ellipsoid.

Obviously, the magnetization is given by

$$I = \vartheta \cos \psi/2, \text{ where } \cos \psi = l_1 l_2 + m_1 m_2 + n_1 n_2.$$

The problem before us is to find the energy of the magnetic elements in the structure *as bounded by an ellipsoidal boundary.* Since we are concerned mainly with the third-order terms, we assume throughout the following discussion that the elements are simply doublets :—

(1) We suppose, in the first place, that the lattice structure instead of being bounded by C, is extended to infinity in all directions ;

(2) Secondly, we suppose throughout all space *outside* the ellipsoid, a homogeneous distribution of moment-density, such as would neutralize the doublets in the extended structure.

The energy of such a system is taken for the energy of our actual system. The error involved is very small except in the immediate neighbourhood of the boundary C. Write $E = E_1 - E'_1$, where

E_1 = energy per unit volume of a system consisting of an infinite doublet-lattice, together with a neutralizing homogeneous moment-density spread through *all* space (even inside the ellipsoid) ; and

E'_1 = energy per unit volume of that part of the neutralizing homogeneous distribution which lies inside the ellipsoid.

To find E_1 :—Consider any one doublet D of lattice L_1. The mutual potential energy of D and the doublets of the L_2-lattice with the *corresponding* neutralizing moment-density may be written in the form,

$$-\mu^2 \left(l_1 \frac{\partial}{\partial x} + m_1 \frac{\partial}{\partial y} + n_1 \frac{\partial}{\partial z}\right)\left(l_2 \frac{\partial}{\partial x} + m_2 \frac{\partial}{\partial y} + n_2 \frac{\partial}{\partial z}\right) \mathrm{V}\,(\mathrm{D}),$$

and the mutual potential energy of D and all the remaining doublets of L_1-lattice together with *its corresponding* neutralizing moment-density is then,

$$-\mu^2 \left(l_1 \frac{\partial}{\partial x} + m_1 \frac{\partial}{\partial y} + n_1 \frac{\partial}{\partial z}\right)^2 \mathrm{U}\,(\mathrm{D})\,;$$

where V (D) = potential produced at D by the system which we get by placing unit charges in place of the doublets at the lattice-points of L_2, together with the corresponding homogeneous neutralizing *density of charge ;* and

U (D) = potential produced at D by the system which we get by placing unit charges at L_1-lattice-points (except the point D), together with its corresponding neutralizing density of charge.

Now actually U and V are infinite. But it is their second derivatives that are involved in the energy-function. And these latter are not infinite. For, consider any one function, say V (D). Divide the system which produces V (D) into a number of parts. Each part satisfied the conditions,

(1) total charge = 0.

(2) moment of the total charge = 0.

Hence the contribution from each part towards V is of the order of r^{-3}.

Therefore the second derivative is of order r^{-5}. And thus if we sum up the contributions of the second derivatives from all the parts into which the original system is divided we get a convergent series.

Briefly stated, the argument comes to this, that although we cannot have

$$V = v_1 + v_2 + v_3 + \dots$$

we can have

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_3}{\partial x^2} + \dots$$

This, in short, is the principle of DE WAARD's method.

[14] *Expression of the energy.*—Returning to the expressions for the energy, the mutual potential energy of a doublet D of L_1 on the one hand and all the others of the L_1 and L_2 on the other is

$$-\mu^2\left[\left(l_1\frac{\partial}{\partial x}+\dots+\right)\left(l_2\frac{\partial}{\partial x}+\dots+\right)V+\left(l_1\frac{\partial}{\partial x}+\dots+\right)^2 U\right].$$

Of this, one half belongs to D itself. Besides the density of doublets in L_1 is ω^{-3}. Hence, the contribution to energy per unit volume from the L_1-doublets in presence of L_2 is,

$$-\frac{\mu^2}{2\,\omega^3}\left[\left(l_1\frac{\partial}{\partial x}+\dots+\right)\left(l_2\frac{\partial}{\partial x}+\dots+\right)V+\left(l_1\frac{\partial}{\partial x}+\dots+\right)^2 U\right].$$

Similarly, the contribution from L_2-doublets is

$$-\frac{\mu^2}{2\,\omega^3}\left[\left(l_1\frac{\partial}{\partial x}+\dots+\right)\left(l_2\frac{\partial}{\partial x}+\dots+\right)V+\left(l_2\frac{\partial}{\partial x}+\dots+\right)^2 U\right].$$

Adding up we get E_1 equal to

$$-\frac{\mu^2}{2\omega^3}\left[\left\{\left(l_1\frac{\partial}{\partial x}+\dots+\right)^2+\left(l_2\frac{\partial}{\partial x}+\dots+\right)^2\right\}U+2\left(l_1\frac{\partial}{\partial x}+\dots+\right)\left(l_2\frac{\partial}{\partial x}+\dots+\right)V\right].$$

It is found that in the case of rectangular lattices the mixed derivatives of U and V vanish, and hence the above formula reduces to

$$E_1=-\frac{\mu^2}{2\,\omega^3}\left[\Sigma\,(l_1^2+l_2^2)\frac{\partial^2 U}{\partial x^2}+2\Sigma\,l_1l_2\frac{\partial^2 V}{\partial x^2}\right],$$

$$=-\frac{\mu^2}{2\,\omega^3}\left[\Sigma\,(l_1^2+l_2^2)\,U_{xx}+2\Sigma\,l_1l_2V_{xx}\right].$$

The calculation of U_{xx}, U_{yy}, V_{xx}, V_{zz}, etc., is postponed for the present.

To find E_1'.—We have always taken our co-ordinate axes along the principal axes of the structure. Let us suppose that the principal axes of the ellipsoidal boundary of C also coincide with these. Now, E_1' it may be recalled, is the energy per unit volume of the ellipsoid due to the neutralizing homogeneous magnetization. This consists of two parts :—

(i) That which neutralizes the doublets of the L_1-lattice, that is to say, a magnetization of intensity $-\mu\,\omega^{-3}$ in the $(l_1\,m_1\,n_1)$ direction ; and

(ii) That which neutralizes the doublets of the L_2-lattice, that is to say, a magnetization of intensity $-\mu\,\omega^{-3}$ in the $(l_2\,m_2\,n_2)$ direction.

Therefore, the total homogeneous magnetization of the ellipsoid has the components

$$I_x=-\mu\,\omega^{-3}\,(l_1+l_2),\quad I_y=-\mu\,\omega^{-3}\,(m_1+m_2),\quad I_z=-\mu\,\omega^{-3}\,(n_1+n_2).$$

The following results may be recalled at this stage.

Let an ellipsoid $\Sigma\,x^2/a^2-1=0$ be magnetized uniformly in the direction $(l_1m_1n_1)$. Then the field created by this—it is demagnetizing—has components $(-lA_0I,\ -mB_0I,\ -nC_0I)$, *i.e.*,

$$(-A_0I_x,\ -B_0I_y,\ -C_0I_z)=H_i,\text{ say,}$$

where

$$A_0=\frac{abc}{2}\int_0^\infty\frac{d\lambda}{(a^2+\lambda)\,\Delta}\ ;\ \text{etc.}$$

and

$$\Delta=\sqrt{(a^2+\lambda)\,(b^2+\lambda)\,(c^2+\lambda)}.$$

The energy of the ellipsoid per unit volume is then

$$-\int_0^I(H_i\,dI)=+\tfrac{1}{2}\,(A_0I_x^2+B_0I_y^2+C_0I_z^2).$$

We have also the general result

$$A_0 + B_0 + C_0 = 1.$$

Using these results in our case we get

$$E_1' = + \mu^2 [(l_1 + l_2)^2 A + (m_1 + m_2)^2 B + (n_1 + n_2)^2 C]/2\omega^3.$$

where

$$A = A_0/\omega^3, \qquad B = B_0/\omega^3, \qquad C = C_0/\omega^3.$$

Thus, finally, the internal energy of the system of doublets bounded by the ellipsoid is obtained in the form $E_1 - E_1'$ per unit volume,

$$E = E_1 - E_1' = - \mu^2 [\Sigma (l_1^2 + l_2^2)(U_{xx} - A) + 2\Sigma l_1 l_2 (V_{xx} - A)]/2\omega^3. \quad . \ (4.1)$$

It will be noticed that in obtaining this expression no account has been taken of temperature effects, and there would be no justification in adopting this to explain all the magnetic phenomena. On general thermo-dynamical grounds RICHARDSON has shown that the magnetization of a body must be a function of H/T. We may, therefore, neglect the effects of temperature (i) in the case of high fields at ordinary temperatures, and (ii) weak fields at rather low temperatures.

[15] *Application to the case of* WEBSTER'S *experiments in connection with iron.*—We have here,

$$\left.\begin{array}{l} U_{xx} = U_{yy} = U_{zz} = \theta, \text{ say} \\ V_{xx} = V_{yy} = V_{zz} = \phi, \text{ say} \end{array}\right\} \text{since the structure is cubic}$$

A = B, in the case of the circular disc, and since in general

$$A + B + C = \omega^{-3}(A_0 + B_0 + C_0) = \omega^{-3},$$

$$C = \omega^{-3} - 2A.$$

Further

$$(l_1, m_1) = (l_2\ m_2) = (l\ m) \quad \text{and} \quad n_1 = n_2 = 0$$

since there is saturation in the plane of the disc.

Making these substitutions in the general expression for E, we get

$$E = - \frac{\mu^2}{2\omega^3}[2l^2(\theta - A) + 2m^2(\theta - A) + 0 + 2l^2(\phi - A) + 2m^2(\phi - A) + 0]$$

$$= - \frac{\mu^2}{\omega^2}(\theta + \phi - 2A)$$

independent of (l, m). Thus E_1, so far as it is due to the doublets, that is the third-order terms, does not involve a variable part of energy. This is why these terms have been left out in the discussion of cubic structures (p. 68).

[16] *Non-cubic structure: Pyrrhotite.*—Suppose that even in this case we consider a circular plate, and apply H in the plane of the disc, so that

$$n' = 0.$$

The structure is hexagonal and consists of two orthorhombic lattices. Let the plate be cut perpendicular to the axis of the hexagonal symmetry. By symmetry the magnetization also is in the plane of the disc. If there is saturation, then

$$(l_1 m_1) = (l_2 m_2) = (lm) \quad \text{and} \quad n_1 = n_2 = 0.$$

We have

$$E = -\frac{\mu^2}{\omega^3}[(U_{xx} + V_{xx} - 2A)\, l^2 + (U_{yy} + V_{yy} - 2A)\, m^2].$$

Notice that $U_{xx} \neq U_{yy}$ and $V_{xx} \neq V_{yy}$ since there is no cubic symmetry.

The energy due to the presence of the external field is

$$-\frac{2\mu}{\omega^3} H\,(ll' + m'm).$$

Thus to search for steady states we have to make stationary,

$$T \equiv (U_{xx} + V_{xx} - 2A)\, l^2 + (U_{yy} + V_{yy} - 2A)\, m^2 + 2H\,(ll' + mm')/\mu,$$

subject to $l^2 + m^2 = 1$.

Following the usual procedure we get

$$\begin{vmatrix} \frac{\partial T}{\partial l} & \frac{\partial T}{\partial m} \\ l & m \end{vmatrix} = 0,$$

i.e.,

$$\frac{H}{\mu}\begin{vmatrix} l' & m' \\ l & m \end{vmatrix} + lm \begin{vmatrix} U_{xx} + V_{xz} - 2A, & U_{yy} + V_{yy} - 2A \\ 1 & 1 \end{vmatrix} = 0.$$

Putting as usual

$$l, m = \cos\psi, \sin\psi\,; \quad l', m' = \cos\phi, \sin\phi\,;$$

we get

$$H \sin(\phi - \psi)/\mu \sin\psi \cos\psi = [(U_{xx} + V_{xx}) - (U_{yy} + V_{yy})].$$

and since ϑ is the saturation value of I or

$$\vartheta = \frac{2}{\omega^3} \cdot \mu,$$

$$H \sin(\phi - \psi)/\vartheta \sin\psi \cos\psi = \tfrac{1}{2}\omega^3 [(U_{xx} + V_{xx}) - (U_{yy} + V_{yy})] \quad . \ . \quad (4.2)$$

This agrees exactly with the result which WEISS obtains from entirely different considerations ; his result is

$$\mathrm{H} \sin (\phi - \psi)/\vartheta \sin \psi \cos \psi = \mathrm{N}_1 - \mathrm{N}_2,$$

where N_1, N_2 are the "constant coefficients" of the molecular field for the axes of x and y respectively.

It is easy to see that if the disc were elliptic and not circular then $\mathrm{A} \neq \mathrm{B}$, and the result we would get is

$$\mathrm{H} \sin (\phi - \psi)/\vartheta \sin \psi \cos \psi = \tfrac{1}{2} \omega^3 [(\mathrm{U}_{xx} + \mathrm{V}_{xx} - 2\mathrm{A}) - (\mathrm{U}_{yy} + \mathrm{V}_{yy} - 2\mathrm{B})].$$

Remarks.—(1) Observe that if we put $\mathrm{U}_{xx} = \mathrm{U}_{yy}$ and $\mathrm{V}_{xx} = \mathrm{V}_{yy}$, the result becomes

$$\mathrm{H} \sin (\phi - \psi)/\vartheta \sin \psi \cos \psi = -\tfrac{1}{2} \omega^3 (2\mathrm{A} - 2\mathrm{B}) = \omega^3 (\mathrm{B} - \mathrm{A}).$$

Therefore the deviation effect would still be accounted for in the case of WEBSTER's experiments, if the plate is of any other form than the circular.

(2) Next, the effect vanishes completely theoretically, if $\mathrm{A} = \mathrm{B}$, *i.e.*, in the circular plate. But actually it *is* present as WEBSTER's experiments show. Hence we have had to give up the simple-doublet conception, and adopt that of an electronic orbit. Then naturally retention of the next non-vanishing terms in the energy-function explains the experimental facts.

PART III.

CHAPTER 5.

Non-saturation States.

[17] An attempt will be made in the following articles to extend the formulæ to the case of weak fields, or to be more precise the case of non-saturation. Among the causes that prevent a body being magnetized to saturation may be mentioned the following : (i) the applied field is not strong enough ; (ii) temperature exerts demagnetizing effects ; (iii) effects of impurities present in the crystal.

As regards impurities we can say that crystals could be obtained in a fairly pure state, and so far as theory is concerned, the effects of the presence of any impurities may be left out. Such, however, is not the case with the temperature-effects. Nevertheless we shall still continue to neglect these in the case of non-saturation, if only to see what formulæ we obtain. We venture to suggest that these will be applicable to experiments made at very low temperatures, and also in cases just under saturation.

Further, in the presence of temperature-effects which set the magnetic elements into agitation and tend to make all directions equally probable, the fundamental assumption of the magnetic elements splitting into two directions will have to be modified. Either the elements will librate about these directions, or rotate about them in a sort of precessional motion. Anyway, the problem ceases to be a statical one, and becomes a dynamical one. Other forms of energy will have to be taken into account in addition to what we have been considering. For these reasons temperature-effects will not be considered in this paper. These limitations probably deprive the analysis in this section of much of its significance. But the formulæ obtained are quite interesting, and are worth being worked out.

Let, as before, the crystal C be in the form of a thin circular plate, and H act in its plane, so that $n' = 0$.

By symmetry the magnetization I will lie in the plane of the disc, and hence

$$n_1 + n_2 = 0.$$

[It is only in the case of saturation that $(l_1\, m_1\, n_1) = (l_2\, m_2\, n_2) = (l\, m\, n)$ with $n = 0$.]

There now seems to be three possible ways in which the magnetic elements could divide themselves into two groups :

I $n_1 = -n_2 = n$, say ; and $(l_1 m_1) = (l_2 m_2) = (lm)$
II $n_1 = -n_2 = 0$; and $(l_1 m_1) \neq (l_2 m_2)$
III $n_1 = -n_2 = n$; and $(l_1 m_1) \neq (l_2 m_2)$.

Obviously III is the most general one and includes the first two. The algebra involved is rather laborious. All the three cases, therefore, will be worked out separately.

[18] Case I $n_1 = -n_2 = n$; and $(l_1 m_1) = (l_2 m_2) = (lm)$.

Obviously the resultant magnetization I is compounded of $\vartheta/2$ along (l, m, n) and $\vartheta/2$ along $(l, m, -n)$. Hence

$$\mathrm{I}_x = \vartheta l, \quad \mathrm{I}_y = \vartheta m, \quad \mathrm{I}_z = 0 \quad \text{and} \quad \mathrm{I} = \vartheta\,(l^2 + m^2)^{\frac{1}{2}} = \vartheta\,(1 - n^2)^{\frac{1}{2}},$$

and the direction of magnetization in the plane of the disc is

$$(l/\sqrt{l^2 + m^2}, \quad m/\sqrt{l^2 + m^2}, \quad 0).$$

So we have,

$$\cos\phi = l', \quad \sin\phi = m' \quad \text{but,}$$
$$\cos\psi = l/\sqrt{l^2 + m^2}, \quad \sin\psi = m/\sqrt{l^2 + m^2}.$$

Now the general expression for the total energy-density is

$$-\tfrac{1}{2}\mu^2\,[\Sigma\,(l_1^2 + l_2^2)\,(\mathrm{U}_{xx} - \mathrm{A}) + 2\Sigma\, l_1 l_2\,(\mathrm{V}_{xx} - \mathrm{A})]/\omega^3 - \mu\mathrm{H}\,[\Sigma\, l'\,(l_1 + l_2)]/\omega^3.$$

Making use of the circumstances in the present case, viz.,

$$(l_1 m_1) = (l_2 m_2) = (lm)\,; \quad n_1 = -n_2 = n, \quad \mathrm{A} = \mathrm{B}, \quad \text{and} \quad \mathrm{C} = \omega^{-3} - 2\mathrm{A}.$$

we get,

$$-\mu^2\,[(p_x + q_x) l^2 + (p_y + q_y)\, m^2 + (p_z - q_z)\, n^2]/\omega^3 - 2\mu\mathrm{H}\,(ll' + mm')/\omega^3,$$

where

$$p_x = U_{xx} - A, \quad p_y = U_{yy} - B, \quad p_z = U_{zz} - C,$$

and

$$q_x = V_{xx} - A, \quad q_y = V_{yy} -, \quad q_z = V_{zz} - C.$$

To make stationary the expression

$$T = (p_x + q_x)\, l^2 + (p_y + q_y)\, m^2 + (p_z - q_z)\, n^2 + 2H\,(ll' + mm')/\mu,$$

subject to

$$l^2 + m^2 + n^2 = 1.$$

We get by the usual method

$$\frac{\partial T}{\partial l}\Big/ l = \frac{\partial T}{\partial m}\Big/ m = \frac{\partial T}{\partial n}\Big/ n,$$

i.e.,

$$[(p_x + q_x)\, l + Hl'/\mu]\, 1/l = [(p_y + q_y)\, m + Hm'/\mu]\, 1/m = (p_z - q_z).$$

i.e.,

$$(p_x + q_x)\, l = (p_z - q_z)\, l - Hl'/\mu,$$

$$(p_y + q_y)\, m = (p_z - q_z)\, m - H\, m'/\mu.$$

whence we get

$$H\,(lm' - l'm)/\mu lm = [(p_x + q_x) - (p_y + q_y)].$$

Substituting the values

$$(l'm') = \cos\phi, \sin\phi \quad \text{and} \quad (l/\sqrt{1 - n^2},\ m/\sqrt{1 - n^2}) = \cos\psi, \sin\psi$$

$$\vartheta = 2\mu\,\omega^{-3} \quad \text{and} \quad I = \vartheta\,(1 - n^2)^{\frac{1}{2}}$$

we get

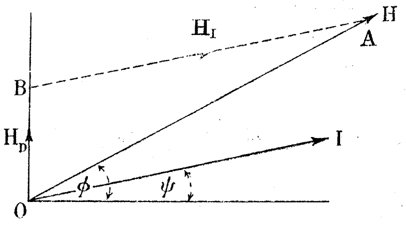
$$\begin{aligned} H \sin(\phi - \psi)/I \cos\psi \sin\psi &= \tfrac{1}{2}\omega^3\, [(p_x + q_x) - (p_y + q_y)] \\ &= \tfrac{1}{2}\omega^3\, [(U_{xx} + V_{xx}) - (U_{yy} + V_{yy})]. \end{aligned}$$

Now this is exactly the same formula as was obtained for the saturation case before, (4.2).

Thus, neglecting temperature-effects the empirical result of Weiss seems to hold for all fields on theoretical grounds. The experimental evidence on this point, however, is not conclusive, For—

(i) So far as Weiss's own experiments on pyrrhotite are concerned, in all his four cases the field strengths employed were strong enough to produce saturation; ($H = 1992$, 4000, 7310, 10275). The way in which he establishes his result is briefly this :

H is decomposed into two vectors H_I and H_D of which H_I is parallel to the magnetization, and H_D parallel to Oy. Thus $\overline{H} = \overline{H_I} + \overline{H_D}$.



Weiss finds in all the four cases, the experimental curves of I sin ψ plotted against H_D are straight lines through the origin. Hence—

$$H_D/I \sin\psi = \text{constant, N, say.}$$

From the triangle O A B, it follows

$$H_D/\sin(\phi - \psi) = H/\sin(\pi/2 + \psi).$$

whence we get

$$H \sin(\phi - \psi)/I \sin\psi \cos\psi = N.$$

But this experimental evidence refers only to high fields—*i.e.*, saturation states.

(ii) In his thesis DE WAARD also remarks: " . . . . volgens WEISS is

$$H \sin(\phi - \psi)/I \sin\psi \cos\psi$$

tennaasteloijy konstant. De afwijkingen blijken groter naarmate men met zwakkere velden te doen heeft" (p. 35). Thus, according to DE WAARD the result does not represent facts when weak fields are employed.

(iii) Thirdly, KUNZ also observes that "in large fields the agreement is good; in weak fields deviations from the straight lines occur which are not yet explained" (p. 171).

Neither DE WAARD, nor KUNZ, give the exact nature of the deviations. We can conclude that the deviations are due possibly to temperature-effects, or if not to temperature-effects, we must at least reject this sort of grouping of the magnetic elements. In the next article we consider the second sort.

[19] Case II. $n_1 = -n_2 = 0$; $(l_1 m_1) \neq (l_2 m_2)$.

Here we have,

$$I_x = \tfrac{1}{2}\vartheta(l_1 + l_2), \quad I_y = \tfrac{1}{2}\vartheta(m_1 + m_2), \quad I_z = 0.$$

$$(l_1 m_1) = (\cos\psi_1, \sin\psi_1); \quad (l_2 m_2) = (\cos\psi_2, \sin\psi_2); \quad 2\psi = \psi_1 + \psi_2.$$

And we easily see that $I = \vartheta \cos\tfrac{1}{2}(\psi_1 - \psi_2)$, and the direction of I is ψ.

The energy-expression in this case becomes:

$$-\mu^2[p_x(l_1^2 + l_2^2) + p_y(m_1^2 + m_2^2) + 0 + 2l_1l_2q_x + 2m_1m_2q_y + 0]/\omega^3$$
$$-\mu H[l'(l_1 + l_2) + m'(m_1 + m_2) + 0]/\omega^3.$$

To make this stationary subject to

$$l_1^2 + m_1^2 = 1; \quad l_2^2 + m_2^2 = 1.$$

we get

$$\frac{\partial T}{\partial l_1}\Big/ l_1 = \frac{\partial T}{\partial m_1}\Big/ m_1$$

and

$$\frac{\partial T}{\partial l_2}\Big/ l_2 = \frac{\partial T}{\partial m_2}\Big/ m_2,$$

i.e.,

$$(l_1p_x + l_2q_x + l'H/\mu)/l_1 = (m_1p_y + m_2q_y + m'H/\mu)/m_1,$$

and

$$(l_2p_x + l_1q_x + l'H/\mu)/l_2 = (m_2p_y + m_1q_y + m'H/\mu)/m_2,$$

so that substituting the values in terms of ψ_1, ψ_2, ϕ we get

$$(p_x - p_y)\sin\psi_1\cos\psi_1 + q_x\cos\psi_2\sin\psi_1 - q_y\sin\psi_2\cos\psi_1 = H\sin(\phi - \psi_1)/\mu$$

and

$$(p_x - p_y)\sin\psi_2\cos\psi_2 + q_x\cos\psi_1\sin\psi_2 - q_y\sin\psi_1\cos\psi_2 = H\sin(\phi - \psi_2)/\mu.$$

Now, by adding and subtracting these we get two results. Subtraction gives

$$\tfrac{1}{2}(p_x - p_y)(\sin 2\psi_1 - \sin 2\psi_2) + (q_x + q_y)\sin(\psi_1 - \psi_2)$$
$$= -2H\sin\tfrac{1}{2}(\psi_1 - \psi_2)\cos(\phi - \psi)/\mu.$$

Now in weak fields, since there is no saturation $\psi_1 \neq \psi_2$, and we can cancel out the factor $\sin(\psi_1 - \psi_2)/2$. We get

$$2(p_x - p_y)\cos\tfrac{1}{2}(\psi_1 - \psi_2)\cos(\psi_1 + \psi_2) + 2(q_x + q_y)\cos\tfrac{1}{2}(\psi_1 - \psi_2)$$
$$= -2H\cos(\phi - \psi)/\mu.$$

i.e.,

$$H\cos(\phi - \psi)/I\,[(q_x + q_y) + (p_x - p_y)\cos 2\psi] = -\mu/\vartheta = -\tfrac{1}{2}\omega^3 \quad . \; . \; . \quad (\xi)$$

since

$$I = \vartheta\cos\tfrac{1}{2}(\psi_1 - \psi_2).$$

Next, addition gives us

$$\tfrac{1}{2}(p_x - p_y)(\sin 2\psi_1 + \sin 2\psi_2) + (q_x - q_y)\sin(\psi_1 + \psi_2)$$
$$= 2H\sin(\phi - \psi)\cos\tfrac{1}{2}(\psi_1 - \psi_2)/\mu.$$

Substituting as above,

$$2\psi = \psi_1 + \psi_2, \qquad \cos\tfrac{1}{2}(\psi_1 - \psi_2) = I/\vartheta,$$
$$\cos(\psi_1 - \psi_2) = 2I^2/\vartheta^2 - 1,$$

we get

$$IH\sin(\phi - \psi)/\sin 2\psi\,[(q_x - q_y)\vartheta^2 + (p_x - p_y)(2I^2 - \vartheta^2)] = \mu/2\vartheta = \tfrac{1}{4}\omega^3. \; . \quad (\eta)$$

Remarks.—I. In the case of weak fields I also is variable. The results (ξ) and (η) thus determine both the magnetization I, and its direction ψ, when the field H and its direction ϕ are given.

II. In obtaining (η), no factor of the types in $\frac{1}{2}(\psi_1 - \psi_2)$ was cancelled. Hence we can use that result for saturation. Putting $I = \vartheta$ we get

$$H\sin(\phi - \psi)/\vartheta\sin 2\psi = \tfrac{1}{4}\omega^3[(q_x - q_y) + (p_x - p_y)],$$

i.e.,

$$H\sin(\phi - \psi)/\vartheta\sin\psi\cos\psi = \tfrac{1}{2}\omega^3[(U_{xx} + V_{xx}) - (U_{yy} + V_{yy})] \text{ as before.}$$

Thus the analysis in this article generalizes WEISS's result for strong fields into the (η) formula for all fields, and it shows in what way deviation would occur in weak fields with a certain kind of "grouping," even if temperature-effects were neglected.

III. In the cubic case $p_x = p_y$; $q_x = q_y$. The formula (η) shows that the deviation-effect must vanish in all fields, on the simple conception of doublets. Hence, even in the case of weak fields, in order to explain the presence of deviation-effects in cubic structure, the fifth-order terms have got to be introduced. In the next article we take up this question.

[20] *Cubic structure: Case III*, $n_1 = -n_2 = n$; $(l_1m_1) \neq (l_2m_2)$.—We make the same supposition as in the last two cases. That is the crystal C is in the form of a circular plate, H acts in the plane of the plate, and by symmetry, magnetization is also in the plane of the disc.

While discussing the saturation case in the first part, we noticed that the form of the energy-function of two magnetic elements was

$$\omega_1 = -\mu^2 P_2/r^3 - \mu^2 a^2 P_4/r^5,$$

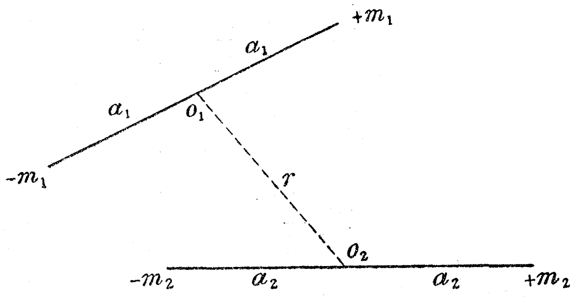
or

$$\omega_2 = -\mu^2 P_2/r^3 + 3\mu^2 a^2 P_4/r^5,$$

according as we adopted the conception of bar-magnets, or permanent electronic orbits.

Now ω_2 differs from ω_1 only in respect of the fifth-order terms. The term in ω_2 is minus three times the fifth-order term in ω_1. No doubt this happens to be the case when the two elements have their axes parallel. But since in either case, the energy must vary continuously as we change the orientation, we infer that even when the axes are not parallel, ω_2 could be deduced from ω_1 by simply multiplying the fifth-order term in ω_1 by minus three. We feel justified in making this short cut as we want to see the form of the solution only. So we proceed to calculate ω_1 :—

Let $2a_1$, $2a_2$ be the lengths of two bar-magnets; $(l_1m_1n_1)$, $(l_2m_2n_2)$ their directions; μ_1, μ_2 their moments; and r the length joining their centres O_1, O_2.



The mutual potential energy of two such bar-magnets is easily worked out as follows.

If s_1, s_2 be lengths measured along the two magnets from O_1, O_2 respectively, then

$$r^{-1} = O_1O_2^{-1} = \text{some function of } s_1s_2$$
$$= \text{say } u(s_1s_2).$$

The mutual potential energy of two poles of the one and the two poles of the other is seen to be

$$m_1 m_2 [u(s_1 - a_1, s_2 - a_2) - u(s_1 - a_1, s_2 + a_2)] \\ - m_1 m_2 [u(s_1 + a_1, s_2 - a_2) - u(s_1 + a_1, s_2 + a_2)].$$

Expanding by TAYLOR's theorem and simplifying this becomes

$$\mu_1 \mu_2 \frac{\partial^2 u}{\partial s_1 \partial s_2} + \frac{\mu_1 \mu_2}{3\,!}\left[a_1^2 \frac{\partial^4 u}{\partial s_1^3 \partial s_2} + a_2^2 \frac{\partial^4 u}{\partial s_1 \partial s_2^3}\right] + \dots .$$

If we put $a_1 = a_2 = \frac{1}{2}a$, $m_1 = m_2$, then $\mu_1 = \mu_2 = \mu = am$, and we get

$$2\omega_1 = \mu^2 \frac{\partial^2 u}{\partial s_1 \partial s_2} + \frac{\mu^2 a^2}{24}\left[\frac{\partial^4 u}{\partial s_1^3 \partial s_2} + \frac{\partial^4 u}{\partial s_1 \partial s_2^3}\right].$$

Hence, modifying the second term we have

$$2\omega_2 = \mu^2 \frac{\partial^2 u}{\partial s_1 \partial s_2} - \frac{\mu^2 a^2}{8}\left(\frac{\partial^4}{\partial s_1^3 \partial s_2} + \frac{\partial^4}{\partial s_1 \partial s_2^3}\right) u.$$

Now we have

$$u^{-2} = r^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

$$\frac{\partial}{\partial s_1} = l_1 \frac{\partial}{\partial x_1} + m_1 \frac{\partial}{\partial y_1} + n_1 \frac{\partial}{\partial z_1},$$

$$\frac{\partial}{\partial s_2} = l_2 \frac{\partial}{\partial x_2} + m_2 \frac{\partial}{\partial y_2} + n_2 \frac{\partial}{\partial z_2},$$

so that

$$\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = 0, \text{ etc.}$$

Write,

$$\frac{\partial u}{\partial x_1} = -\frac{\partial u}{\partial x_2} = \frac{\partial u}{\partial x}.$$

We have

$$2\omega_2 = -\mu^2 \left(\Sigma\, l_1 \frac{\partial}{\partial x}\right)\left(\Sigma\, l_2 \frac{\partial}{\partial x}\right) u \\ + \frac{\mu^2 a^2}{8}\left[\left(\Sigma\, l_1 \frac{\partial}{\partial x}\right)^3 \left(\Sigma\, l_2 \frac{\partial}{\partial x}\right) + \left(\Sigma\, l_1 \frac{\partial}{\partial x}\right)\left(\Sigma\, l_2 \frac{\partial}{\partial x}\right)^3\right] u,$$

neglecting terms of order higher than u^5.

Now, $\Sigma\, 2\omega_2$ represents the contribution to the energy per cell from the mutual action between the two lattices L_1, L_2. To this we must add the contribution arising from each lattice by itself. Obviously these contributions are $[\omega_2]_{l_2 = l_1}$, and $[\omega_2]_{l_1 = l_2}$.

In other words, if we write explicitly $\omega_2 = \omega_2(l_1, l_2)$, the energy per unit volume of the system formed by L_1, L_2 interlocked is

$$W_2 = \frac{1}{\omega^3}\left[\sum_{L_1} \omega_2(l_1, l_1) + 2 \sum_{L_1, L_2} \omega_2(l_1, l_2) + \sum_{L_2} \omega_2(l_2, l_2)\right].$$

Out of this expression, the third-order terms are

$$-\frac{\mu^2}{2\,\omega^3}\left[\sum_{L_1}\left(l_1\frac{\partial}{\partial x}+\ldots+\right)^2+2\sum_{L_1,\,L_2}\left(l_1\frac{\partial}{\partial x}+\ldots+\right)\left(l_2\frac{\partial}{\partial x}+\ldots+\right)+\sum_{L_2}\left(l_2\frac{\partial}{\partial x}+\ldots+\right)^2\right]u.$$

The summations over the infinite lattice of the mixed derivatives of u vanish since the lattice is rectangular, and as before the third-order terms reduce to

$$-\frac{\mu^2}{2\,\omega^3}[\Sigma\,(l_1^2+l_2^2)\,(\mathrm{U}_{xx}-\mathrm{A})+2\Sigma\,l_1 l_2\,(\mathrm{V}_{xx}-\mathrm{A})].$$

The remaining terms are of the fifth order and they are,

$$+\frac{\mu^2 a^2}{8}\left[\sum_{L_1}\frac{\partial^4}{\partial s_1^{\,4}}+\sum_{L_2}\frac{\partial^4}{\partial s_2^{\,4}}+\sum_{L_1,\,L_2}\left\{\frac{\partial^4}{\partial s_1^{\,3}\,\partial s_2}+\frac{\partial^4}{\partial s_1\,\partial s_2^{\,3}}\right\}\right]u.$$

We know that the space-centred cubic structure is specified by

$$x=ps,\quad y=qs,\quad z=rs,\quad \text{where}\quad p=q=r\ (\text{mod. }2).$$

And in $\sum\limits_{L_1}$ or $\sum\limits_{L_2}$, if one magnet be taken at the origin all the others will have $(p,\,q,\,r)$ even. But in $\sum\limits_{L_1,\,L_2}$ if a magnet of L_1 is at the origin, those of L_2 will have $(p,\,q,\,r)$ odd.

Hence the above expression may be written as

$$\frac{\mu^2 a^2}{8\,\omega^3}\left[\sum_{pqr}\left(\frac{\partial^4}{\partial s_1^{\,4}}+\frac{\partial^4}{\partial s_2^{\,4}}\right)u+\sum_{p'q'r'}\left(\frac{\partial^4}{\partial s_1^{\,3}\,\partial s_2}+\frac{\partial^4}{\partial s_1\,\partial s_2^{\,3}}\right)u'\right],$$

where

$(p,\,q,\,r)$ take even values from $-\infty$ to $+\infty$,

and

$(p',\,q',\,r')$ take odd values from $-\infty$ to $+\infty$,

of course

$$u^{-2}=x^2+y^2+z^2=s^2\,(p^2+q^2+r^2),$$
$$u'^{-2}=x^2+y^2+z^2=s^2\,(p'^2+q'^2+r'^2).$$

On account of cubic symmetry we must have

$$\sum_{pqr}\frac{\partial^{r+s+t}}{\partial x^r\,\partial y^s\,\partial z^t}\,u=0,$$

when at least one of the integers r, s, t equals zero.* This result is true whether $(p\,q\,r)$ take *all* integer values, all odd values only, or all even values only. We shall make use of this in what follows.

Write for the present

$$l'_1=l_1\frac{\partial}{\partial x},\quad m'_1=m_1\frac{\partial}{\partial y},\quad n'_1=n_1\frac{\partial}{\partial z},$$

$$l'_2=l^2\frac{\partial}{\partial x},\quad m'_2=m_2\frac{\partial}{\partial y},\quad n'_2=n_2\frac{\partial}{\partial z}.$$

---

* The temporary duplicated uses of r and s here must not be confused.

Then,

$$\frac{\partial^4}{\partial s_1{}^4} + \frac{\partial^4}{\partial s_2{}^4} = (l'_1 + m'_1 + n'_1)^4 + (l'_2 + m'_2 + n'_2)^4.$$

Now, in our grouping $n_1 = -n_2 = n$.

Therefore

$$\frac{\partial^4}{\partial s_1{}^4} + \frac{\partial^4}{\partial s_2{}^4} = (l'_1 + m'_1 + n'_1)^4 + (l'_2 + m'_2 - n'_1)^4$$
$$= (l'^4_1 + m'^4_1 + l'^4_2 + m'^4_2 + 2n'^4_1) + 6\,(l'^2_1 m'^2_1 + l'^2_2 m'^2_2) + 12 n'^2_1\,(l'^2_1 + m'^2_1 + l'^2_2 + m'^2_2).$$

leaving out terms which lead to vanishing series.

Now, write

$$\sum_{pqr}{}' \frac{\partial^4 u}{\partial x^4} = h_1, \qquad \sum_{pqr}{}' \frac{\partial^4 u}{\partial y^4} = h_2, \qquad \sum_{pqr}{}' \frac{\partial^4 u}{\partial z^4} = h_3,$$
$$\sum_{p'q'r'}{}' \frac{\partial^4 u'}{\partial x^4} = h'_1, \qquad \sum_{p'q'r'} \frac{\partial^4 u'}{\partial y^4} = h'_2, \qquad \sum_{p'q'r'} \frac{\partial^4 u'}{\partial z^4} = h'_3,$$
$$\sum_{pqr} \frac{\partial^4 u}{\partial y^2\,\partial z^2} = j_1, \qquad \sum_{pqr} \frac{\partial^4 u}{\partial z^4\,\partial x^2} = j_2, \qquad \sum_{pqr} \frac{\partial^4 u}{\partial x^2\,\partial y^2} = j_3.$$
$$\sum_{p'q'r'} \frac{\partial^4 u'}{\partial y^2\,\partial z^2} = j'_1, \qquad \sum_{p'q'r'} \frac{\partial^4 u'}{\partial z^2\,\partial x^2} = j'_2, \qquad \sum_{p'q'r'} \frac{\partial^4 u'}{\partial x^2\,\partial y^2} = j'_3.$$

where $(p\,q\,r)$ take all even values and $(p'q'r')$ take all odd values.

Cubic symmetry at once gives

$h_1 = h_2 = h_3 = h$, say, $j_1 = j_2 = j_3 = j$, $h'_1 = h'_2 = h'_3 = h'$, and $j'_1 = j'_2 = j'_3 = j'$

Making use of these we get

$$\sum_{pqr} \left(\frac{\partial^4 u}{\partial s_1{}^4} + \frac{\partial^4 u}{\partial s_2{}^4}\right) = h\,(l_1^4 + m_1^4 + l_2^4 + m_2^4 + 2n^4)$$
$$+ j\,[6\,(l_1^2 m_1^2 + l_2^2\,m_2^2) + n^2\,(l_1^2 + m_1^2 + l_2^2 + m_2^2)],$$

using $\Sigma\, l_1^2 = \Sigma\, l_2^2 = 1$ and omitting the constant part we get

$$2\,(3j - h)\,(l_1^2 m_1^2 + l_2^2 m_2^2) + 4\,(3j - h)\,(n^2 - n^4) = \text{say } Q_5.$$

To find, now,

$$\sum_{p'q'r'} \left[\frac{\partial^4}{\partial s_1{}^3\,\partial s_2} + \frac{\partial^4}{\partial s_1\,\partial s_2{}^3}\right] u.$$

Here we have to simplify,

$$(l'_1 + m'_1 + n'_1)^3\,(l'_2 + m'_2 + n'_2) + (l'_1 + m'_1 + n'_1)\,(l'_2 + m'_2 + n'_2)^3.$$

Omitting terms involving an odd power, as these lead to vanishing summations, we get

$$(l'_1 l'_2 + m'_1 m'_2 + n'_1 n'_2)\,(l'^2_1 + m'^2_1 + n'^2_1 + l'^2_2 + m'^2_2 + n'^2_2)$$
$$+ 2\,\Sigma\,[(l'_1 m'_1 + l'_2 m'_2)\,(l'_2 m'_1 + l'_1 m'_2)],$$

i.e.,

$$[\Sigma\, l_1 l_2 (l_1^2 + l_2^2)] \frac{\partial^4 u'}{\partial x^4} + \Big[\Sigma\, \{l_1 l_2 (m_1^2 + n_1^2 + m_2^2 + n_2^2) + \Sigma\, \{(l_1 m_1 + l_2 m_2)(l_2 m_1 + l_1 m_2)\}\Big] \frac{\partial^4 u'}{\partial x^2\, \partial y^2}.$$

Using the relations $\Sigma\, l_1^2 = \Sigma\, l_2^2 = 1$, $n_1 = -n_2 = n$ the above reduces to, say R_5, where

$$\begin{aligned} R_5 = {} & [l_1 l_2 (l_1^2 + l_2^2) + m_1^2 m_2^2 (m_1^2 + m_2^2)\, h' - 2h' n^4 \\ & + [l_1 l_2 (m_1^2 + m_2^2) + m_1 m_2 (l_1^2 + l_2^2) + 2 (l_1 m_1 + l_2 m_2)(l_1 m_2 + l_2 m_1)]\, j' \\ & + 4j' (l_1 l_2 + m_1 m_2) n^2 - 6j' n^2 (1 - n^2). \end{aligned}$$

Thus the energy per unit volume arising from the fifth-order terms is

$$\mu^2 a^2 (Q_5 + R_5)/8\omega^3.$$

Also, the energy arising from the third-order terms has been seen to be

$$-\frac{\mu^2}{2\,\omega^3} P_3, \quad \text{where}$$

$$P_3 = \Sigma\, (l_1^2 + l_2^2)(U_{xx} - A) + 2\, \Sigma\, l_1 l_2 (V_{xx} - A).$$

In the present case of a circular plate of cubic structure,

$$U_{xx} = U_{yy} = U_{zz}, \qquad V_{xx} = V_{yy} = V_{zz},$$

$$A = B, \quad C = \omega^{-1} - 2A \quad \text{and} \quad n_1 = -n_2 = n.$$

Hence

$$P_3 = (l_1^2 + l_2^2)\, p_x + (m_1^2 + m_2^2)\, p_y + (n_1^2 + n_2^2)\, p_z + 2 l_1 l_2 q_x + 2 m_1 m_2 q_y + 2 n_1 n_2 q_z.$$

Where as before $p_x = U_{xx} - A$, etc., and $q_x = V_{xx} - A$, etc. These, of course, are constants. Further $p_x = p_y$, $q_x = q_y$; as $A = B$. Hence,

$$P_3 = p_x (l_1^2 + m_1^2 + l_2^2 + m_2^2) + 2n^2 p_z + 2 (l_1 l_2 + m_1 m_2)\, q_x - 2n^2 q_z.$$

Writing

$$p_x = p_y = p, \quad q_x = q_y = q, \quad \text{we have} \quad p_z = p + A - C, \quad q_z = q + A - C.$$

Hence P_3 reduces to $2p + 2n^2 (A - C) + 2 (l_1 l_2 + m_1 m_2)\, q - 2n^2 (q + A - C)$.

Omitting the constant part this becomes simply $2 (l_1 l_2 + m_1 m_2) q - 2n^2\, q$.

Further, energy arising from the external field H is

$$-\frac{H\mu}{\omega^3} [l' (l_1 + l_2) + m' (m_1 + m_2) + 0].$$

Collecting together all the terms and results we make the total energy $-\omega^3 T/\mu^2$ stationary subject to $\Sigma\, l_1^2 = 1$, $\Sigma\, l_2^2 = 1$. And T is equal to

$$\begin{aligned} & q\,[l_1 l_2 + m_1 m_2 - n^2] \\ & \quad - \frac{a^2}{8}[2\,(3j - h)\,(l_1^2 m_1^2 + l_2^2 m_2^2) + h'\,\{l_1 l_2\,(l_1^2 + l_2^2) + m_1 m_2\,(m_1^2 + m_2^2)\} \\ & \qquad + j'\,\{l_1 l_2\,(m_1^2 + m_2^2) + m_1 m_2\,(l_1^2 + l_2^2) + 2\,(l_1 m_1 + l_2 m_2)\,(l_2 m_1 + l_1 m_2)\} \\ & \qquad + 4j'\,(l_1 l_2 + m_1 m_2)\,n^2 - 6j'n^2\,(1 - n^2) + 4\,(3j - h)\,n^2\,(1 - n^2) - 2h'n^4] \\ & \quad - \frac{\mathrm{H}}{\mu}\,[l'\,(l_1 + l_2) + m'\,(m_1 + m_2)]. \end{aligned}$$

As usual we get two relations

$$\frac{\partial \mathrm{T}}{\partial l_1}\Big/ l_1 = \frac{\partial \mathrm{T}}{\partial m_1}\Big/ m_1 \quad \ldots\ldots\ldots\ldots\ldots \quad (5.1)$$

$$\frac{\partial \mathrm{T}}{\partial l_2}\Big/ l_2 = \frac{\partial \mathrm{T}}{\partial m_2}\Big/ m_2. \quad \ldots\ldots\ldots\ldots\ldots \quad (5.2)$$

Adding (5.1) and (5.2) we get one relation. Call it ξ-relation.

Subtraction gives say η-relation.

These are much too complicated, and we shall work them out in the next article for the particular case of $n = 0$.

[21] $\left[\dfrac{\partial \mathrm{T}}{\partial l_1}\Big/ l_1 = \dfrac{\partial \mathrm{T}}{\partial m_1}\Big/ m_1\right]$ gives,

$$\begin{aligned} & q\,(l_2 m_1 - l_1 m_2) \\ & \quad - \frac{a^2}{8}[4\,(3j - h)\,l_1 m_1\,(m_1^2 - l_1^2) + h'\,\{(m_1 l_2^3 - l_1 m_2^3) + 3 l_1 m_1\,(l_1 l_2 - m_1 m_2)\} \\ & \qquad + j'\,\{l_2 m_1\,(m_1^2 + m_2^2) - m_2 l_1\,(l_1^2 + l_2^2) + 2 l_1 m_1\,(m_1 m_2 - l_1 l_2) \\ & \qquad\quad + 2\,(l_1 m_1 + l_2 m_2)\,(m_1 m_2 - l_1 l_2) + 2\,(l_1 m_2 + l_2 m_1)\,(m_1^2 - l_1^2) \\ & \qquad\qquad + 4j'n^2\,(l_2 m_1 - l_1 m_2)\}] \\ & \quad + \frac{\mathrm{H}}{\mu}\,(l' m_1 - l_1 m') = 0. \end{aligned}$$

The relation $\left[\dfrac{\partial \mathrm{T}}{\partial l_2}\Big/ l_2 = \dfrac{\partial \mathrm{T}}{\partial m_2}\Big/ m_2\right]$ is got by interchanging $(l_1 m_1)$ with $(l_2 m_2)$ in the above relation.

We shall put $n = 0$ and work out the result further.

Adding up the two results after putting $n = 0$ gives the result—

$$\begin{aligned} & \frac{a^2}{8}[4\,(h - 3j)\,\{l_1 m_1\,(l_1^2 - m_1^2) + l_2 m_2\,(l_2^2 - m_2^2)\} \\ & \quad + h'\,\{(l_2^2 - m_1^2)\,(m_1 l_2 + l_2 m_1) + 3\,(l_1 m_1 + l_2 m_2)\,(l_1 l_2 - m_1 m_2)\} \\ & \quad + j'\,\{(m_1^2 + m_2^2)\,(l_2 m_1 + l_1 m_2) - (l_1^2 + l_2^2)\,(l_2 m_1 + l_1 m_2) \\ & \qquad + 6\,(l_1 m_1 + l_2 m_2)\,(m_1 m_2 - l_1 l_2) + 2\,(l_1 m_2 + l_2 m_1)\,(m_1^2 + m_2^2 - l_1^2 - l_2^2)\}] \\ & \qquad\qquad = \frac{\mathrm{U}}{\mu}[(l' m_1 - l_1 m') + (l' m_2 - l_2 m')]. \quad \ldots \quad (\xi)\text{-relation.} \end{aligned}$$

Similarly, subtraction gives

$$
\begin{aligned}
2q\,(l_2m_1 - l_1m_2) & \\
-\frac{a^2}{8}\Big[4\,(h-3j)\,\{(l_1^2-m_1^2)\,l_1m_1 - (l_2^2-m_2^2)\,l_2m_2\} & \\
+h'\,\{(l_2m_1-l_1m_2)+(l_2m_1+l_1m_2)\,(m_1^2-m_2^2)+3\,(l_1m_1-l_2m_2)\,(l_1l_2-m_1m_2)\} & \\
+j'\,\{(m_1^2+m_2^2)\,(l_2m_1-l_1m_2)+(l_1^2+l_2^2)\,(l_2m_1-l_1m_2) & \\
+2\,(m_1m_2-l_1l_2)\,(l_1m_1-l_2m_2)+2\,(l_1m_2+l_2m_1)\,(m_1^2-l_1^2-m_2^2-l_2^2)\}\Big] & \\
+\frac{\mathrm{H}}{\mu}\,[(l'm_1-l_1m')-(l'm_2-l_2m')]=0. \quad . \quad . \quad . \quad (\eta)\text{-relation.} &
\end{aligned}
$$

Reduce these ξ and η-relations by introducing the usual notation $l', m' = \cos\phi, \sin\phi$; $l_1, m_1 = \cos\psi_1, \sin\psi_1$; $l_2, m_2 = \cos\psi_2, \sin\psi_2$; $2\psi = \psi_1 + \psi_2$ and $\mathrm{I}/\vartheta = \cos\frac{1}{2}(\psi_1 - \psi_2)$; write also $f_n(\mathrm{I}/\vartheta) = \cos n\,(\psi_1 - \psi_2)$.

Take first the ξ-relation:

$$
\begin{aligned}
\text{Coefficient of } (h-3j) &= 2\sin 2\psi_1 \cos 2\psi_1 + 2\sin 2\psi_2 \cos 2\psi_2 = \sin 4\psi_1 + \sin 4\psi_2 \\
&= 2\sin 2\,(\psi_1+\psi_2)\cos 2\,(\psi_1-\psi_2) = 2\sin 4\psi\,.\,f_2\,(\mathrm{I}/\vartheta).
\end{aligned}
$$

$$
\begin{aligned}
\text{Coefficient of } h' &= (\cos^2\psi_1 + \cos^2\psi_2 - 1)\sin(\psi_1+\psi_2) \\
&\qquad + \tfrac{3}{2}\,(\sin 2\psi_1 + \sin 2\psi_2)\cos(\psi_1+\psi_2), \\
&= \tfrac{1}{2}\,(\cos 2\psi_1 + \cos 2\psi_2)\sin 2\psi + 3\sin 2\psi\cos(\psi_1-\psi_2)\cos 2\psi \\
&= \cos 2\psi\cos(\psi_1-\psi_2)\sin 2\psi + 3\sin 2\psi\cos(\psi_1-\psi_2)\cos 2\psi \\
&= 4\cos 2\psi\sin 2\psi\cos(\psi_1-\psi_2) = 2\sin 4\psi\,f_1\,(\mathrm{I}/\vartheta).
\end{aligned}
$$

$$
\begin{aligned}
\text{Coefficient of } j' &= \sin(\psi_1+\psi_2)\,(\cos 2\psi_1 + \cos 2\psi_2) + 6\cos(\psi_1+\psi_2)\,(\tfrac{1}{2}\sin 2\psi_1 \\
&\qquad + \tfrac{1}{2}\sin 2\psi_2) + 2\sin(\psi_1+\psi_2)\,(\cos 2\psi_1 + \cos 2\psi_2) \\
&= 3\sin 2\psi\,(\cos 2\psi_1 + \cos 2\psi_2) + 6\cos 2\psi + \sin(\psi_1+\psi_2)\cos(\psi_1-\psi_2) \\
&= 6\sin 4\psi\cos(\psi_1-\psi_2) = 6\sin 4\psi\,f_1\,(\mathrm{I}/\vartheta).
\end{aligned}
$$

Hence we get

$$
\frac{a^2}{8}\,[(h-3j)\,2\sin 4\psi f_2 + 2h' f_1 \sin 4\psi - 6j' f_1 \sin 4\psi] = -\frac{2\mathrm{H}}{\mu}\sin(\phi-\psi)\,\mathrm{I}/\vartheta,
$$

i.e.,

$$
\frac{\mathrm{HI}\sin(\phi-\psi)}{\sin 4\psi\,[(hf_2 + h'f_1) - 3\,(jf_2 + j'f_1)]} = -\frac{a^2\mu}{8}\,\vartheta = -\tfrac{1}{16}a^2\,\omega^3\vartheta^2.
$$

Take now the η-relation:

$$
\text{Coefficient of } q = 2\sin(\psi_1-\psi_2) = 4\sin\tfrac{1}{2}\,(\psi_1-\psi_2)\,\mathrm{I}/\vartheta.
$$

$$
\begin{aligned}
\text{Coefficient of } (h-3j) &= 2\sin 2\psi_1\cos 2\psi_1 - 2\sin 2\psi_2\cos 2\psi_2 \\
&= 8\sin\tfrac{1}{2}\,(\psi_1-\psi_2)\,\mathrm{I}/\vartheta\cos 4\psi.
\end{aligned}
$$

$$\text{Coefficient of } h' = \sin(\psi_1 - \psi_2) + \sin(\psi_1 + \psi_2)(\sin^2\psi_1 - \sin^2\psi_2) + \tfrac{3}{2}(\sin 2\psi_1 - \sin 2\psi_2)\cos(\psi_1 + \psi_2)$$

$$= \sin(\psi_1 - \psi_2)\{1 + \sin^2 2\psi + 3\cos^2 2\psi\}$$

$$= 4\,\mathrm{I}/\vartheta \sin\tfrac{1}{2}(\psi_1 - \psi_2)(1 + \cos^2 2\psi)$$

$$\text{and coefficient of } j' = 2\sin(\psi_1 - \psi_2) - \cos(\psi_1 + \psi_2)(\sin 2\psi_1 - \sin 2\psi_2) + 2\sin(\psi_1 + \psi_2)(\cos 2\psi_2 - \cos 2\psi_1)$$

$$= 2\sin(\psi_1 - \psi_2)\{1 - \cos^2 2\psi + 2\sin^2 2\psi\}$$

$$= \sin(\psi_1 - \psi_2)\{2 - 2\cos^2 2\psi + 4\sin^2 2\psi\}$$

$$= \sin(\psi_1 - \psi_2)\, 6\sin^2 2\psi$$

$$= 6\sin\tfrac{1}{2}(\psi_1 - \psi_2)\cos\tfrac{1}{2}(\psi_1 + \psi_2)(1 - \cos 4\psi)$$

$$= 6\sin\tfrac{1}{2}(\psi_1 - \psi_2)(1 - \cos 4\psi)\,\mathrm{I}/\vartheta.$$

$$\text{and the coefficient of } \mathrm{H}/\mu = \sin(\psi_1 - \phi) - \sin(\psi_2 - \phi)$$

$$= 2\sin\tfrac{1}{2}(\psi_1 - \psi_2)\cos(\phi - \psi).$$

Thus we get

$$\frac{\mathrm{I}}{\vartheta}\sin\tfrac{1}{2}(\psi_1 - \psi_2)\left[4q - \frac{a^2}{8}\{8(h - 3j)f_1\cos 4\psi + 4h'(1 + \cos^2 2\psi) + 6j'(1 - \cos 4\psi)\}\right] = -\frac{2\mathrm{H}}{\mu}\sin\tfrac{1}{2}(\psi_1 - \psi_2)\cos(\phi - \psi).$$

Since there is no saturation $\psi_1 \neq \psi_2$; hence cancelling out the factor $\sin\tfrac{1}{2}(\psi - \psi_2)$ we have

$$\frac{\mathrm{H}\cos(\phi - \psi)}{\mathrm{I}\left[2q - \frac{a^2}{8}\{4(h - 3j)f_1\cos 4\psi + 2h'(1 + \cos^2 2\psi) + 3j'(1 - \cos 4\psi)\}\right]} = -\frac{\mu}{\vartheta} = -\tfrac{1}{2}\omega^3.$$

Summing up the results we have in the case of a space-centred cubic structure, with weak fields, H acting in the plane of a circular plate, with the assumption of

$$(l_1 m_1) \neq (l_2 m_2) \quad \text{and} \quad n_1 = -n_2 = 0,$$

the following results :

$$\mathrm{H}\cos(\phi - \psi)\Big/\mathrm{I}\left[2q - \frac{a^2}{8}\{4(h - 3j)f_1\cos 4\psi + 2h'(1 + \cos^2 2\psi) + 3j'(1 - \cos 4\psi)\}\right] = -\tfrac{1}{2}\omega^3,$$

and

$$\mathrm{HI}\sin(\phi - \psi)/\sin 4\psi\,[(hf_2 + h'f_1) - 3(jf_2 + j'f_1)] = -\tfrac{1}{16}a^2\vartheta^2\omega^3.$$

[22] Some observations on the results obtained in the last article.

Consider the result

$$\mathrm{HI}\sin(\phi - \psi)/\sin 4\psi\,[(hf_2 + h'f_1) - 3(jf_2 + j'f_1)] = -\tfrac{1}{16}a^2\vartheta^2\omega^3.$$

In obtaining this, no factor of the type $\sin \frac{1}{2}(\psi_1 - \psi_2)$ was cancelled out; we can therefore apply it in the case of saturation. To show that this reduces exactly to the result in the first part, put $I = \vartheta$, $\psi_1 = \psi_2$, and thus

$$f_n = \cos n(\psi_1 - \psi_2) = 1.$$

The formula becomes

$$H \sin(\phi - \psi)/\vartheta \sin 4\psi = -\tfrac{1}{16} a^2 \omega^3 [(h + h') - 3(j + j')].$$

Now,

$$h + h' = \sum_{pqr} \frac{\partial^4}{\partial x^4}\left(\frac{1}{r}\right), \quad j + j' = \sum_{pqr} \frac{\partial^4}{\partial x^2\, \partial y^2}\left(\frac{1}{r}\right), \quad [p = q = r \ (\mathrm{mod.}\ 2)].$$

Observe that, putting $u^{-2} = r^2 = x^2 + y^2 + z^2$,

$$\frac{\partial u}{\partial x} = -xu^3,$$

$$\frac{\partial^2 u}{\partial x^2} = -u^3 + 3x^2u^5,$$

$$\frac{\partial^3 u}{\partial x^3} = 3xu^5 + 6xu^5 - 15x^3u^7 = 9xu^5 - 15x^3u^7,$$

and

$$\frac{\partial^4 u}{\partial x^4} = 9u^5 - 45x^2u^7 - 45x^2u^7 + 105x^4u^9,$$

$$\frac{\partial^3 u}{\partial x^2\, \partial y} = 3yu^5 - 15x^2yu^7,$$

and

$$\frac{\partial^4 u}{\partial x^2 \partial y^2} = 3u^5 - 15y^2u^7 - 15x^2u^7 + 105x^2y^2u^9.$$

Hence we have

$$(h + h') - 3(j + j') = \sum_{pqr} [45y^2u^7 - 45x^2u^7 + 105(x^4 - 3x^2y^2)u^9]$$

$$= 105\left[\sum_{p,q,r} \frac{x^4}{r^9} - \sum_{pqr} \frac{x^2y^2}{r^9}\right].$$

$$\left(\text{Since by cubic symmetry } \sum_{pqr} \frac{y^2}{r^7} = \sum_{pqr} \frac{x^2}{r^7}.\right)$$

And since $x, y, z = ps, qs, rs$, we have $u^{-2} = s^2(p^2 + q^2 + r^2)$. And

$$(h + h') - 3(j + j') = 105(P - 3Q)/s^5,$$

where P, Q are the same constants as introduced in the first part, viz.,

$$P = \sum_{pqr} \frac{p^4}{(p^2 + q^2 + r^2)^{9/2}}, \qquad p = q = r \ (\mathrm{mod.}\ 2)$$

$$Q = \sum_{pqr} \frac{q^2r^2}{(p^2 + q^2 + r^2)^{9/2}}.$$

Thus the formula reduces to

$$\mathrm{H} \sin (\phi - \psi)/\vartheta \sin \psi \cos \psi \cos 2\psi = 105\, a^2 \omega^3 (3\mathrm{Q} - \mathrm{P})/4s^5$$

as expected.

Next consider the other result

$$\mathrm{H} \cos (\phi - \psi)\Big/\mathrm{I}\left[2q - \frac{a^2}{8}\{4(h - 3j) f_1 \cos 4\psi + \ldots\}\right] = -\tfrac{1}{2}\omega^3.$$

If we adopt a simple doublet conception and neglect, therefore, the fifth-order terms, we must put $a = 0$, and we get

$$\mathrm{H} \cos (\phi - \psi)/2q\mathrm{I} = -\tfrac{1}{2}\omega^3.$$

Now the result obtained before for the non-cubic case was

$$\mathrm{H} \cos (\phi - \psi)/\mathrm{I}\,[q_x + q_y - (p_x - p_y) \cos 2\psi] = -\tfrac{1}{2}\,\omega^3.$$

In the cubic case $p_x = p_y$; $q_x = q_y = q$ and hence this reduces to the above result.

3. If we confine our attention to the magnetization produced when H acts along a principal axis, we get

$$\mathrm{H}/\mathrm{I} = -q\omega^3.$$

Now $q = \mathrm{V}_{xx} - \mathrm{A} = \mathrm{V}_{yy} - \mathrm{A}$.

By cubical symmetry,

$$\mathrm{V}_{xx} = \mathrm{V}_{yy} = \mathrm{V}_{zz} = \tfrac{1}{3}\omega^{-3}\text{*}$$

$$\mathrm{A} + \mathrm{B} + \mathrm{C} = \omega^{-3}(\mathrm{A}_0 + \mathrm{B}_0 + \mathrm{C}_0) = \omega^{-3},$$

$$\mathrm{A} = \mathrm{B}, \qquad \mathrm{C} = \omega^{-3} - 2\,\mathrm{A}.$$

Further

$$\mathrm{A}_0 = \tfrac{1}{2}abc\int_0^\infty \frac{d\lambda}{(a^2 + \lambda)\,\Delta}, \text{ etc.,}$$

and hence

$$\mathrm{A}_0 < \mathrm{C}_0, \text{ if } a > c,$$

which is the case for our circular plate where $a = b >> c$. In any case A_0† $<< \frac{1}{3}$ and thus $\mathrm{A} << \frac{1}{3}\omega^{-3}$. Hence q is positive.

Thus the result $\mathrm{H}/\mathrm{I} = -\omega^3 q$ leads to a negative value for the susceptibility

$$\chi = \mathrm{I}/\mathrm{H} = -q^{-1}\omega^{-3} = -(\tfrac{1}{3} - \mathrm{A}_0)^{-1}.$$

The explanation of the negative sign is probably this. In a crystal, when temperature effects are neglected, the stable state is the saturation state, and χ loses its meaning. To pass on to any other state than the saturation state, we must apply H in the opposite direction, and naturally χ comes out negative.

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* Appendix C, note 2.

† Appendix C, note 3.

APPENDIX A.

The energy of two elementary current circuits.

We consider the case of two equal circuits in parallel planes, circular in form, each with a radius b less than the distance between their centres.

The energy of either due to the field of the other is $\omega = \frac{1}{2}i\mathrm{N}$, where N is the flux through the circuit S of the magnetic field created by the other. Obviously,

$$\omega = \tfrac{1}{2}i\mathrm{N} = -\iint_{\mathrm{S}} \frac{\partial \Omega}{\partial z}\, d\mathrm{S}.$$

Now,

$$\frac{\partial}{\partial z} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial\mu}, \quad \mu = \cos\theta.$$

The usual formula for Ω the magnetic potential of a circular current* is

$$\Omega = 2\pi i\left[\tfrac{1}{2}\frac{b^2}{r^2}\mathrm{P}_1 - \frac{1\cdot 3}{2\cdot 4}\frac{b^4}{r^4}\mathrm{P}_3 \ldots + (-1)^{n+1}\frac{1\,.\,3\,.\,5\,.\,7 \ldots 2n-1}{2\,.\,4\,.\,6 \ldots 2n}\mathrm{P}_{2n-1}\ldots\right]$$

for $r > b$.

Hence,

$$-\frac{\partial\Omega}{\partial r} = 2\pi i\left[\frac{b^2}{r^3}\mathrm{P}_1 - \tfrac{3}{2}\frac{b^4}{r^5}\mathrm{P}_3 \ldots\right]$$

and

$$-\frac{\partial\Omega}{\partial\mu} = -2\pi i\left[\tfrac{1}{2}\frac{b^2}{r^2}\mathrm{P}'_1 - \tfrac{3}{8}\frac{b^4}{r^4}\mathrm{P}'_3 \ldots\right].$$

so that

$$-\frac{\partial\Omega}{\partial z} = 2\pi i\left[\frac{b^2}{r^3}\mathrm{P}_1 - \tfrac{3}{2}\frac{b^4}{r^5}\mathrm{P}_3\right]\mu + \frac{\sin^2\theta}{r}2\pi i\left[\tfrac{1}{2}\frac{b^2}{r^2}\mathrm{P}'_1 - \tfrac{3}{8}\frac{b^4}{r^4}\mathrm{P}'_3\right]$$

$$= 2\pi i\left[\frac{b^2}{r^3}\left(\mu\mathrm{P}_1 - \frac{\sin^2\theta}{2}\mathrm{P}'_1\right) + \frac{b^4}{r^5}\left(\tfrac{3}{8}\sin^2\theta\mathrm{P}'_3 - \tfrac{3}{2}\mu\mathrm{P}_3\right)\right]$$

Now,

$$\mu\mathrm{P}_1 - \tfrac{1}{2}\sin^2\theta\mathrm{P}'_1 = \mu^2 - \tfrac{1}{2}(1-\mu^2) = \tfrac{1}{2}(3\mu^2 - 1) = \mathrm{P}_2$$

and, since

$$\mathrm{P}_3(\mu) = \tfrac{1}{2}(5\mu^3 - 3\mu)$$

$$\tfrac{1}{4}\sin^2\theta\mathrm{P}'_3 - \mu\mathrm{P}_3 = \tfrac{1}{4}(1-\mu^2)\tfrac{1}{2}(15\mu^2 - 3) - \tfrac{1}{2}\mu(5\mu^2 - 3\mu)$$

$$= \tfrac{1}{8}(-3 + 30\mu^2 - 35\mu) = -\mathrm{P}_4(\mu).$$

Thus we have

$$-\frac{\partial\Omega}{\partial z} = \frac{2\pi i b^2}{r^3}\mathrm{P}_2 - \frac{3\pi i b^4}{r^4}\mathrm{P}_4, \text{ neglecting higher terms.}$$

We have now to integrate this expression over the area S, and express the result in terms of (R, ϑ) the co-ordinates of the centre of S.

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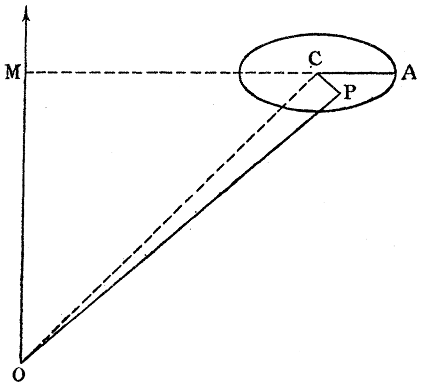
* JEANS 'Electricity and Magnetism,' p. 422.

It is not difficult to see that so far as the fifth-order term is concerned we need simply write R and cos ϑ for r and cos θ respectively and multiply by the area S which is πb^2. The error involved will be of a higher order than the fifth. Thus

$$-\iint_s \frac{3\pi i b^4}{r^3} \mathrm{P}_4 (\cos\theta)\, d\mathrm{S} = -\frac{3\pi^2 i b^6}{\mathrm{R}^5} \mathrm{P}_4 (\cos\vartheta) \quad \text{correct to } \mathrm{R}^{-5}.$$

To consider now

$$\iint_s \frac{2\pi i b^2}{r^3} \mathrm{P}_2 (\cos\theta)\, d\mathrm{S}.$$



$$\mathrm{M\hat{O}C} = \vartheta,$$
$$\mathrm{OC} = \mathrm{R},$$
$$\mathrm{M\hat{O}P} = \theta,$$
$$\mathrm{OP} = r,$$
$$\mathrm{A\hat{C}P} = \phi,$$
$$\mathrm{CP} = \rho,$$

OM || the axis of the circuit.

The fig. shows at once that

$$\cos\theta = \frac{\mathrm{R}}{r}\cos\vartheta$$

$$r^2 = \mathrm{R}^2 + \rho^2 + 2\mathrm{R}\rho \sin\vartheta\cos\phi,$$

$$d\mathrm{S} = \rho\, d\rho\, d\phi$$

Now,

$$\iint \frac{2\pi i b^2}{r^3} \mathrm{P}_2 (\cos\theta)\, d\mathrm{S} = 2\pi i b^2 \iint \frac{\mathrm{P}_2}{r^3}\, d\mathrm{S}.$$

The integrand

$$= \frac{\mathrm{P}_2 (\cos\theta)}{r^3} = \tfrac{1}{2}[3\mathrm{R}^2 \cos^2\vartheta/(\mathrm{R}^2 + \rho^2 + 2\mathrm{R}\rho\sin\vartheta\cos\phi)^{5/2}$$
$$- (\mathrm{R}^2 + \rho^2 + 2\mathrm{R}\rho\sin\vartheta\cos\phi)^{3/2}]$$

$$= \tfrac{1}{2}\left[\frac{3\cos^2\vartheta}{\mathrm{R}^3}\left\{1 - \tfrac{5}{2}\left(\frac{\rho^2 + 2\mathrm{R}\rho\sin\vartheta\cos\phi}{\mathrm{R}^2}\right) + \frac{5\cdot 7}{2\cdot 4}\left(\frac{\rho^2 + 2\mathrm{R}\rho\sin\vartheta\cos\phi}{\mathrm{R}^2}\right)\cdots\right\}\right.$$
$$\left. - \frac{1}{\mathrm{R}^3}\left\{1 - \tfrac{3}{2}\left(\frac{\rho^2 + 2\mathrm{R}\rho\sin\vartheta\cos\phi}{\mathrm{R}^2}\right) + \frac{3\cdot 5}{2\cdot 4}\left(\frac{\rho^2 + 2\mathrm{R}\rho\sin\vartheta\cos\phi}{\mathrm{R}^2}\right)^2\cdots\right\}\right]$$

$$= \tfrac{1}{2}\left[\frac{3\cos^2\vartheta - 1}{\mathrm{R}^3} - \frac{2\rho\sin\vartheta\cos\phi}{\mathrm{R}^4}(\tfrac{5}{2}\,.\,3\cos^2\vartheta - \tfrac{3}{2}) - \frac{\rho^2}{\mathrm{R}^5}(\tfrac{5}{2}\,.\,3\cos^2\vartheta - \tfrac{3}{2})\right.$$
$$\left. + \frac{4\rho^2\sin^2\vartheta\cos^2\phi}{\mathrm{R}^5}\left(\frac{5\cdot 7}{2\cdot 4}3\cos^2\vartheta - \frac{3\cdot 5}{2\cdot 4}\right)\right].$$

Each of the terms in the above expression is to be multiplied by the areal element $\rho d\rho d\phi$ and integrated over the range $\int_0^b d\rho \int_0^{2\pi} d\phi$.

Observing that $\int_0^b \int_0^{2\pi} \rho \, d\rho \, d\phi = \pi b^2$, $\int_0^b \int_0^{2\pi} \rho^2 \, d\rho \cos \phi \, d\phi = 0$, $\int_0^b \int_0^{2\pi} \rho^3 \, d\rho \, d\phi = \frac{1}{2}\pi b^4$,

and $\int_0^b \int_0^{2\pi} \rho^3 \, d\rho \cos^2 \phi \, d\phi = \frac{1}{4}\pi b^4$,

we get the result of integration as

$$\frac{1}{2}\left[\frac{3\cos^2\vartheta - 1}{\mathrm{R}^3}\pi b^2 - 0 - \left(\tfrac{15}{2}\cos^2\vartheta - \tfrac{3}{2}\right)\tfrac{1}{2}\pi b^4 \frac{1}{\mathrm{R}^5} \right.$$
$$\left. + \frac{4}{2\cdot 4}\sin^2\vartheta\,(105\cos^2\vartheta - 15)\,\tfrac{1}{4}\frac{\pi b^4}{\mathrm{R}^5}\right]$$
$$= \frac{\pi b^2}{2\mathrm{R}^3}(3\cos^2\vartheta - 1) + \frac{\pi b^4}{16\mathrm{R}^5}\, 3\,[2\,(1 - \cos^2\vartheta) + \sin^2\vartheta\,(35\cos^2\vartheta - 5)]$$
$$= \mathrm{R}^{-3}\pi b^2 \mathrm{P}_2(\cos\vartheta) - \tfrac{3}{16}\pi b^4\,[3 - 30\cos^2\vartheta + 35\cos^4\vartheta]\cdot \mathrm{R}^{-5}$$
$$= \frac{\pi b^2}{\mathrm{R}^3}\mathrm{P}_2(\cos\vartheta) - \frac{3\pi b^4}{2\mathrm{R}^5}\mathrm{P}_4(\cos\vartheta).$$

Thus taking account of the results obtained hitherto we get

$$\omega = \tfrac{1}{2}i\mathrm{N} = -\iint_s \frac{\partial\Omega}{\partial z}\, d\mathrm{S} = \iint_s \left[\frac{2\pi i}{r^3}\mathrm{P}_2 - \frac{3\pi i b^4}{r^5}\mathrm{P}_4\right] d\mathrm{S}$$
$$= -\frac{3\pi^2 i^2 b^6}{2\mathrm{R}^5}\mathrm{P}_4 + \frac{\pi^2 i^2 b^4}{\mathrm{R}^3}\mathrm{P}_2 - \frac{3}{2}\frac{\pi^2 i^2 b^6}{\mathrm{R}^5}\mathrm{P}_4 = \frac{\lambda^2}{r^3}\mathrm{P}_2 - \frac{3\lambda^2 b^2}{r^5}\mathrm{P}_4,$$

where λ $(= i\pi b^2)$ is the moment of the equivalent magnetic shell, *i.e.*, the magnetic moment of the circuit.

Now if the currents are permanent, in the sense that they are not subject to induction, the energy must be taken as, $-\frac{1}{2}i\mathrm{N}$. And in that case we have

$$\omega = -\lambda^2\mathrm{P}_2/r^3 + 3\lambda^2 b^2 \mathrm{P}_4/r^5.$$

APPENDIX B.

Calculation of P, Q where

$$\mathrm{P} = \sum_{pqr} \frac{p^4}{(p^2 + q^2 + r^2)^{9/2}}, \quad \mathrm{Q} = \sum_{pqr} \frac{q^2 r^2}{(p^2 + q^2 + r^2)^{9/2}}, \qquad [p \equiv q \equiv r \ (\text{mod. } 2)].$$

We have

$$3\,(\mathrm{P} + 2\mathrm{Q}) = \sum_{pqr} \frac{1}{(p^2 + q^2 + r^2)^{5/2}} = 3^{-s/2}\mathrm{B}_s \text{ where } s = 5.$$

Now the values of B_s, from $s = 4$ to $s = 30$ have been calculated by J. E. Jones and A. E. Ingham.* Thus $P + 2Q$ is known. In what follows we shall calculate $P - Q$. The values of P, Q can then obviously be easily deduced,

$$P - Q = \tfrac{1}{2} \sum_{pqr} \frac{(q^2 - r^2)^2}{(p^2 + q^2 + r^2)^{9/2}} . \qquad [p = q = r \text{ (mod. 2)}]$$

Separating the terms in which $(p\,q\,r)$ are even and those in which they are odd, we write

$$P - Q = \tfrac{1}{64} \sum_{1}^{\infty} \frac{W_n}{n^{9/2}} + \tfrac{1}{2} \sum_{n = 3 \text{ (mod. 8)}} \frac{W_n}{n^{9/2}} . \quad \ldots \ldots \ldots \quad (1)$$

where

$$W_n = \sum_{l_1^2 + l_2^2 + l_3^2 = n} (l_2^2 - l_3^2)^2 .$$

We shall calculate the sum of the first twenty terms in the first series and three terms in the second. Denoting it by S_1, we have

$$P - Q > S_1 .$$

To find an upper bound, we make use of the simple fact that $(q^2 - r^2)^2 \leq (p^2 + q^2 + r^2)^2$.

Thus $P - Q < S_1 +$ (the remaining terms with $(p^2 + q^2 + r^2)^2$ written in place of $(q^2 - r^2)^2$).

That is to say we have

$$P - Q < S_1 + \tfrac{1}{2} \sum_{pqr} \frac{1}{(p^2 + q^2 + r^2)^{9/2}} - \tfrac{1}{64} \sum_{1}^{20} \frac{W'_n}{n^{5/2}} - \tfrac{1}{2} \sum \frac{W'_n}{n^{5/2}} , \quad \ldots \quad (2)$$

where W'_n = number of representations of n as a sum of three squares.

Write the above as

$$P - Q < S_1 + \tfrac{1}{2}\, 3^{-5/2}\, B_5 - M - N .$$

Thus we have

$$S_1 < (P - Q) < S_1 + \tfrac{1}{2}\, 3^{-5/2} \,.\, B_5 - (M + N) . \quad \ldots \ldots \quad (3)$$

Before giving the calculations of S_1, M, N, we shall state the results that will be obtained.

$$S_1 = 0{\cdot}08839^{82}_{93}$$

$$M = 0{\cdot}15735^{496}_{794}$$

$$N = 0{\cdot}2982^{50}_{82}$$

$$B_5 = 14{\cdot}7585^{07}_{13} . \quad \ldots \ldots \quad \text{(Jones and Ingham)}$$

Whence,

$$P + 2Q = \tfrac{1}{3}\, 3^{-5/2} B_5 = 3^{-7/2} B_5 = 3^{-7/2} \times 14{\cdot}7585^{07}_{13} = 0{\cdot}31557 . \quad \ldots \quad (4)$$

Next, $\tfrac{1}{2}\, 3^{-5/2} B_5 = 0{\cdot}4733^{6}_{7}$.

Hence substituting we get

$$0{\cdot}08839^{82}_{93} < (P - Q) < 0{\cdot}08839^{82}_{93} + 0{\cdot}4733^{6}_{7} - (M + N) .$$

i.e.,

$$0{\cdot}08839^{82}_{93} < P - Q < 0{\cdot}1061^{2924}_{4324} \quad \ldots \ldots \ldots \quad (5)$$

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* 'Roy. Soc. Proc.,' A., vol. 107, p. 640 (1925).

We shall solve now (4) and (5) as two simultaneous equations, and get the limits for P, Q.

$$P + 2Q = 0{\cdot}31557 \qquad\qquad P + 2Q = 0{\cdot}31557$$

$$P - Q = 0{\cdot}08839 \qquad\qquad P - Q = 0{\cdot}10615$$

Therefore, Therefore,

$$P = 0{\cdot}1641\tfrac{1}{2} \qquad\qquad P = 0{\cdot}1759\tfrac{5}{6}$$

$$Q = 0{\cdot}0757\tfrac{2}{3} \qquad\qquad Q = 0{\cdot}0698\tfrac{0}{1}$$

$$\text{and } 3Q - P = 0{\cdot}0630\tfrac{6}{7} \qquad\qquad \text{and } 3Q - P = 0{\cdot}0334\tfrac{6}{7}$$

Thus the results of this appendix are

$$0{\cdot}16411 < P < 0{\cdot}17596$$

$$0{\cdot}06980 < Q < 0{\cdot}07573$$

$$0{\cdot}03346 < 3\,Q - P < 0{\cdot}06307.$$

Calculation of S_1, M, N :—

Write, $t_n = W_n/n^{9/2}$, $t'_n = 1/n^{5/2}$.

Then

$$S_1 = \tfrac{1}{64}\sum_{1}^{20} t_n + \tfrac{1}{2}(t_3 + t_{11} + t_{19} + t_{27})\,;$$

$$M = \tfrac{1}{64}\sum_{1}^{20} W'_n t'_n,$$

$$N = \tfrac{1}{2}(W'_3 t'_3 + W'_{11} t'_{11} + W'_{19} t'_{19} + W'_{27} t'_{27}).$$

The calculation of each term was effected by means of a logarithmic table, and the results are entered in the table attached. We shall indicate here how W_n and W'_n are to be found for any given n. There does not appear to exist any formula for these. Each case has to be separately worked out. We shall exhibit the procedure in a typical case. Take $n = 17$. Then as our notation goes,

$$W_{17} = \sum_{l_1^2+l_2^2+l_3^2=17} (l_2^2 - l_3^2)^2,$$

and W'_{17} = number of representations in which 17 can be expressed as a sum of three squares.

It is easily seen that $17 = 0^2 + 1^2 + 4^2 = 2^2 + 2^2 + 3^2$ and that there is no other way possible.

Consider the case $(l_1 l_2 l_3) = (2, 2, 3)$. Here (2, 2, 3) may have the numbers with either the plus or minus sign. So eight cases arise $(\pm 2, \pm 2, \pm 3)$. Now the number may again change places, *e.g.* $(\pm 3, \pm 2, \pm 2)$ and $(\pm 2, \pm 3, \pm 2)$. So in this case the different cases become in all 24. If all were different there would be 48 different combinations of them. Thus the relation

$$17 = 2^2 + 2^2 + 3^2$$

gives to W'_{17} a contribution 24.

Next, when

$$(l_1 l_2 l_3) = (\pm 3, \ \pm 2, \ \pm 2)$$
$$l_2^{\ 2} - l_3^{\ 2} = 0.$$

And each of the remaining 16 combinations gives $(l_2^{\ 2} - l_3^{\ 2})^2 = (3^2 - 2^2)^2 = 25$. Therefore the contribution to W_{17} is $25 \times 16 = 400$.

Now, consider

$$17 = 0^2 + 1^2 + 4^2$$

We have

$$(l_1 l_2 l_3) = (0, \ \pm 1, \ \pm 4), (\pm 1, 0, \ \pm 4), (\pm 1, \ \pm 4, 0).$$

Again the numbers 4, 1 may be interchanged, everything remaining the same. Then we get 24 combinations. Next,

from $$(0, \ \pm 1, \ \pm 4), \qquad \Sigma \, (l_2^{\ 2} - l_3^{\ 2})^2 = 15^2 \times 4,$$

and since we can interchange 1 and 4 we get $15^2 \times 4 \times 2 = 1800$.

Similarly from

$$(\pm 1, 0, \ \pm 4), \ \Sigma \, (l_2^{\ 2} - l_3^{\ 2})^2 = 16^2 \times 4 \times 2 = 2048,$$

and from

$$(\pm 4, \ \pm 1, 0), \ \Sigma \, (l_2^{\ 2} - l_3^{\ 2})^2 = 1 \times 8 = 8.$$

So finally we get

$$W_{17} = 400 + 1800 + 2048 + 8 = 4256,$$

and

$$W'_{17} = 24 + 24 = 48.$$

The whole working may be represented as follows :—

n.	$l_1 l_2 l_3$.	$(l_2^{\ 2} - l_3^{\ 2})^2$.	W_n.	W'_n.
17	0 1 4 0 −1 4 four rows	$15^2 \times 4 \times 2$ $= 1800$	1800	24
	1 0 4 1 0 −4 four rows	$16^2 \times 4 \times 2$ $= 2048$	2048	
	4 1 0 4 −1 0 four rows	$1 \times 4 \times 2$ $= 8$	8	
	2 2 3 2 −2 3 eight rows	$5^2 \times 8 \times 2$ $= 400$	400	24
		Total, 4256	Total, 4256	48

RESULTS OF CALCULATIONS.

n.	$t_n = \frac{W_n}{n^{9/2}}$.	$t'_n = \frac{1}{n^{5/2}}$.	W_n.	W'_n.
1	$4 \cdot 00$	$1 \cdot 000$	4	6
2	$0 \cdot 3535^{5}_{6}$	$0 \cdot 1767^{7}_{8}$	8	12
3	0	$0 \cdot 06415^{0}_{1}$	0	8
4	$0 \cdot 1250$	$0 \cdot 03125$	64	6
5	$0 \cdot 1488^{3}_{4}$	$0 \cdot 01788^{8}_{9}$	208	24
6	$0 \cdot 04536^{0}_{1}$	$0 \cdot 01134^{0}_{1}$	144	24
7	0	—	0	0
8	$0 \cdot 01104^{8}_{9}$	$0 \cdot 005524^{2}_{3}$	128	12
9	$0 \cdot 02377^{6}_{7}$	$0 \cdot 004115^{2}_{3}$	468	30
10	$0 \cdot 03693^{5}_{6}$	$0 \cdot 003162$	1168	24
11	$0 \cdot 02108^{7}_{8}$	$0 \cdot 002491^{8}_{9}$	1024	24
12	0	$0 \cdot 002004^{6}_{7}$	0	8
13	$0 \cdot 009477^{7}_{8}$	$0 \cdot 001641^{1}_{2}$	976	24
14	$0 \cdot 01090^{8}_{9}$	$0 \cdot 001363^{5}_{6}$	1568	48
15	0	—	0	0
16	$0 \cdot 003906^{2}_{3}$	$0 \cdot 0009765^{6}_{7}$	1024	6
17	$0 \cdot 01235^{8}_{9}$	$0 \cdot 0008392^{2}_{3}$	4256	48
18	$0 \cdot 009538^{0}_{1}$	$0 \cdot 0007274^{7}_{8}$	4248	36
19	$0 \cdot 001802^{6}_{7}$	$0 \cdot 0006355^{0}_{1}$	1024	24
20	$0 \cdot 004651^{0}_{1}$	$0 \cdot 0005590^{1}_{2}$	3328	24
27	$0 \cdot 003337^{3}_{4}$	$0 \cdot 00026^{3}_{4}$	9216	32

The reader will verify from the foregoing table that,

$$\sum_{1}^{20} t_n = 4 \cdot 8182^{275}_{550},$$

$$t_3 + t_{11} + t_{19} + t_{27} = 0 \cdot 02622^{69}_{81}.$$

Thus

$$\begin{aligned} S_1 &= \tfrac{1}{64}\,[4\cdot8182^{275}_{550}] + \tfrac{1}{2}\,[0\cdot02622^{69}_{81}] \\ &= 0\cdot07528^{48}_{52} + 0\cdot01311^{34}_{41} \\ &= 0\cdot08839^{82}_{93}. \end{aligned}$$

$$\begin{aligned} M &= \tfrac{1}{64}\sum_{1}^{20} W'_n\, t'_n \\ &= \tfrac{1}{32}\,[12\,\{t'_5 + t'_6 + t'_{10} + t'_{11} + t'_{13} + 2t'_{14} + 2t'_{17} + t'_{19} + t'_{20}\} \\ &\qquad + 3\,\{t'_1 + 2t'_2 + t'_4 + 2t'_8 + 5t'_9 + t'_{16} + 6t'_{18}\} + 4\,\{t'_3 + t'_{12}\}] \\ &= \tfrac{3}{8}\,X + \tfrac{3}{32}\,Y + \tfrac{1}{8}\,Z, \end{aligned}$$

$$X = 0\cdot04212^{285}_{529}$$

$$Y = 1\cdot4217^{5578}_{7665}$$

$$Z = 0\cdot06615^{46}_{57}.$$

Thus $M = 0\cdot015796^{06}_{98} + 0\cdot1332^{8960}_{9156} + 0\cdot008269^{3}_{4} = 0\cdot15735^{496}_{794}.$

Similarly—

$$\begin{aligned} N &= 4\,(t'_3 + 3t'_{11} + 3t'_{19} + 4t'_{27}) \\ &= 0\cdot2982^{5000}_{8212}. \end{aligned}$$

Summary.—The Results are—

$$0\cdot16411 < P < 0\cdot17596$$

$$0\cdot06980 < Q < 0\cdot07573$$

$$0\cdot03346 < 3Q - P < 0\cdot06307.$$

We require the sign and magnitude of (3Q — P) and can safely say that

$$0\cdot032 < 3Q - P < 0\cdot064$$

i.e., (3Q — P) is essentially positive and of the order 10^{-2}.

APPENDIX C.

Note I.—Mr. INGHAM has pointed out an alternative way of calculating (Q — P). It is based on a paper by INGHAM and JONES on " The Calculation of certain crystal potential constants, etc."* It will be now sketched out briefly :—

We have—

$$P - Q = \tfrac{1}{2}\sum_{pqr}\frac{(q^2 - r^2)^2}{(p^2 + q^2 + r^2)^{s/2}} \text{ where } s = 9.$$

It is easily seen that (P — Q) equals

$$\tfrac{1}{2}\,\frac{4}{(s-4)\,(s-2)}\left[\frac{d^2}{d\alpha^2}\sum\frac{1}{\{p^2 + (1+\alpha)\,q^2 + (1+\alpha)\,r^2\}^{(s-4)/2}}\right]_{\alpha=0}$$

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* ' Roy. Soc. Proc.,' *loc. cit.*

The sum in the bracket can be transformed by formulæ given by INGHAM and JAMES in their paper and the differentiations performed without any theoretical difficulty, if we suppose, as we must for convergence, that $s > 7$. It becomes convenient to split the sum into two parts, viz., one in which $(p\,q\,r)$ take even values and one in which they take all odd values. Considering for simplicity the part in which even values are taken, put

$$p = 2l_1, \quad q = 2l_2, \quad r = 2l_3, \quad \text{so that} \quad l_1 l_2 l_3$$

take *all* integer values. We have now to evaluate

$$2^{3-s}\left[\frac{d^2}{d\alpha^2}\,\frac{4}{(s-4)(s-2)}\,\Sigma'\,\frac{1}{\{l_1^2+(1+\alpha)\,l_2^2+(1-\alpha)\,l_3^2\}^{(s-4)/2}}\right]_{\alpha=0}.$$

The dash meaning the combination $l_1 = l_2 = l_3 = 0$ is omitted. We have now the Epstein ζ-function corresponding to a quadratic form with $D \equiv 1 - \alpha^2$; to this we apply Lemma 1 in the paper quoted above, and perform the operation $\left[\frac{d^2}{d\alpha^2}\right]_{\alpha=0}$.

The first term in the right hand side of 4·11 in the paper yields—

$$\sum_{l_1^2+l_2^2+l_3^2<x} \frac{(l_2^2+l_3^2)^2}{(l_1^2+l_2^2+l_3^2)^{(s-4)/2}} = \sum_{n<x} \frac{W_n}{n^{(s-4)/2}},$$

where

$$W_n = \sum_{l_1^2+l_2^2+l_3^2=n} (l_2^2 - l_3^2)^2.$$

In the second term the only part depending on α is $D^{-\frac{1}{2}}$, and so there is no trouble there. The third term disappears altogether. In the fourth we have to evaluate $\left[\frac{d^2}{d\alpha^2} Q_\nu(x)\right]_{\alpha=0}$. This is equal to

$$\begin{aligned}\nu(\nu-1) &\sum_{l_1^2+l_2^2+l_3^2<x} (l_2^2-l_3^2)^2\,(x-l_1^2-l_2^2-l_3^2)^{\nu-2}\\ &= \nu(\nu-1)\sum_{n<x} W_n\,(x-n)^{\nu-2}\\ &= \nu(\nu-1)\,\omega_{\nu-2}(x), \text{ say.}\end{aligned}$$

The terms which have to be computed actually, are not therefore very different from the corresponding terms in the function A_s in that paper. The only difference is that in place of a_n the coefficients are W_n; these can naturally be calculated so far as required without much difficulty, and we have calculated them in the last Appendix.

The real trouble appears to be with the error-term $R_k(x, s)$. Here we are concerned with $\left[\frac{d^2}{d\alpha^2} Q_k(x)\right]_{\alpha=0}$ that is, in virtue of (4.05) with

$$\Sigma\left[\frac{d^2}{d\alpha^2}\,\frac{J_{k+3/2}\left(2\pi\sqrt{l_1^2+\dfrac{l_2^2}{1+\alpha}+\dfrac{l_3^2}{1-\alpha}}\right)}{\sqrt{1-\alpha^2}\left(l_1^2+\dfrac{l_2^2}{1-\alpha}+\dfrac{l_3^2}{1-\alpha}\right)^{\frac{1}{2}k+3/4}}\right].$$

This differentiates out into a combination of Bessel Functions and we get functions of the type which INGHAM and JONES denote by $J_\mu(\rho, X)$, so that some use could be made of the inequalities in Lemmas 3 and 5. The formulæ would no doubt be complicated.

The second part corresponding to (pqr), taking odd values could be dealt with in the same way as in the B_s series in the paper; the odd terms cancelling out.

It is clear that some sort of approximation could be got in this way; but it is impossible to say without working out the error terms what degree of accuracy would be attained.

Note II.—Values of U_{xx}, V_{xx} for the cubic structure are easily obtained :—

By definition,

V = potential produced at a point of say the L_2-lattice, by the system which consists of unit charges at the L_1-lattice-points with the corresponding neutralizing homogeneous charge density.

This latter homogeneous density $= -\omega^{-3}$. Hence by POISSON's equation—

$$\nabla^2 V = +\omega^{-3}, \text{ LORENTZ' mixed units.}$$

But in the cubic symmetry,

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial z^2},$$

and hence each $= \frac{1}{3}\omega^{-3}$.

Similarly, $U_{xx} = U_{yy} = U_{zz} = \frac{1}{3}\omega^{-3}$.

Note III.—The following may be quoted from MAXWELL's "Electricity and Magnetism." In his notation

$$(L, M, N) \equiv (-4\pi A_0, -4\pi B_0, -4\pi C_0).$$

"In the case of a very flattened planetoid N becomes in the limit equal to -4π, and L and M become $-c\pi^2/a$." Thus in our case, $A_0 = \pi c/4a$, where c is the thickness of the plate and a the diameter of the plate. Taking the date of WEBSTER's experiments, in the notation we have adopted.

$$\pi^2 c/a = 0\cdot771 \quad \ldots\ldots \quad \text{Disc A,}$$

and

$$\pi^2 c/a = 0\cdot734 \quad \ldots\ldots \quad \text{Disc B.}$$

Hence dividing by 4π we get obviously $A_0 << \frac{1}{3}$.